
**BTZ extensions of globally
hyperbolic singular flat
spacetimes**

Abstract

Minkowski space, namely \mathbb{R}^3 endowed with the quadratic form $-dt^2 + dx^2 + dy^2$, is the local model of 3 dimensionnal flat spacetimes. Recent progress in the description of globally hyperbolic flat spacetimes showed strong link between Lorentzian geometry and Teichmüller space. We notice that Lorentzian generalisations of conical singularities are useful for the endeavours of describing flat spacetimes, creating stronger links with hyperbolic geometry and compactifying spacetimes. In particular massive particles and extreme BTZ singular lines arise naturally. This paper is three-fold. First, prove background properties which will be useful for future work. Second generalise fundamental theorems of the theory of globally hyperbolic flat spacetimes. Third, defining BTZ-extension and proving it preserves Cauchy-maximality and Cauchy-completeness.

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1 Introduction

1.1 Context and motivations

The main interest of our study are *singular flat globally hyperbolic Cauchy-complete spacetimes*. This paper is part of a longer-term objective : construct correspondances between spaces of hyperbolic surfaces, singular spacetimes and singular Euclidean surfaces. A central point which underlies this entire paper as well as a following to come is as follows.

Starting from a compact surface Σ with a finite set S of marked points, the Teichmüller space of (Σ, S) is the set of complete hyperbolic metric on $\Sigma \setminus S$ up to isometry. The universal cover of a point of the Teichmüller space is the Poincaré disc \mathbb{H}^2 which embeds in the 3-dimensional *Minkowski space*, denoted by $\mathbb{E}^{1,2}$. Namely, Minkowski space is \mathbb{R}^3 endowed with the indefinite quadratic form $-dt^2 + dx^2 + dy^2$ where (t, x, y) are the cartesian coordinates of \mathbb{R}^3 and the hyperbolic plane embeds as the quadric $\{-t^2 + x^2 + y^2 = -1, t > 0\}$. It is in fact in the cone $C = \{t > 0, -t^2 + x^2 + y^2 < 0\}$ which direct isometry group is exactly the group of isometry of the Poincaré disc : $\text{SO}_0(1, 2)$. A point of Teichmüller space can then be described as a representation of the fundamental group $\pi_1(\Sigma \setminus S)$ in $\text{SO}_0(1, 2)$ which image is a lattice Γ .

If the set of marked points is trivial, $S = \emptyset$, then the lattice Γ is uniform. The hyperbolic surface \mathbb{H}^2/Γ embeds into C/Γ giving our first non trivial examples of flat globally hyperbolic Cauchy-compact spacetimes.

If on the contrary, the set of marked point is not trivial, $S \neq \emptyset$, then the lattice Γ contains parabolic isometries each of which fixes point-wise a *null* ray on the boundary of the cone C . The cusp of the hyperbolic metric on $\Sigma \setminus S$ correspond bijectively to the equivalence classes of these null rays under the action of Γ . More generally, take a discrete subgroup Γ of $\text{SO}_0(1, 2)$. The group Γ may have an elliptic isometries *i.e.* a torsion part. Therefore, on the one hand \mathbb{H}^2/Γ is a complete hyperbolic surface with conical singularities. On the other hand, C/Γ is a flat spacetime with a Lorentzian analogue of conical singularities : *massive particles*. This spacetime admits a connected sub-surface which intersects exactly once every rays from the origin in the cone C . Furthermore, this sub-surface is naturally endowed with a riemannian metric with respect to which it is complete.

As an example, consider the modular group $\Gamma = \text{PSL}(2, \mathbb{Z})$. A fundamental domain of the action of $\text{PSL}(2, \mathbb{Z})$ on \mathbb{H}^2 is decomposed into two triangles isometric to the same ideal hyperbolic triangle T of angles $\frac{\pi}{2}$ and $\frac{\pi}{3}$. The surface \mathbb{H}^2/Γ is then obtained by gluing edge to edge these two triangles (see [Rat] for more details about the modular group). The *suspension* of the hyperbolic triangle T is $\text{Susp}(T) = \mathbb{R}_+^* \times T$ with the metric $-dt^2 + t^2 ds_T^2$. It can be realised as a cone of triangular basis in Minkowski space as shown on figure 1.a. An edge of the triangulation of \mathbb{H}^2/Γ corresponds to a face of one of the two suspensions, then the suspensions can be glued together face to face accordingly. In this way we obtain this way a flat spacetime but the same way the vertices of the triangulation give rise to conical singularities, the vertical edges will give rise to singular lines in our spacetime. There are three singular lines we can put in two categories following the classification of Barbot, Bonsante and Schlenker [BBS11, BBS14].

- Two *massive particles* going through the conical singularities of \mathbb{H}^2/Γ . The corresponding vertical edges are endowed with a negative-definite semi-riemannian metric.
- One *extreme BTZ-line* toward which the cusp of \mathbb{H}^2/Γ seems to tend like in figure 1.b. The corresponding vertical edge is endowed with a null semi-riemannian metric.

The spacetime C/Γ can be recovered by taking the complement of the extreme BTZ-line. Still, we constructed something more which satisfies two interesting properties.

- Take a horizontal plane in Minkowski space above the origin. It intersects $\text{Susp}(T)$ along a Euclidean triangle. The gluing of the suspensions induces a gluing of the corresponding Euclidean triangles. We end up with a *polyhedral surface* which intersects exactly once every rays from the origin : our singular spacetime with extreme BTZ-line have a polyhedral *Cauchy-surface*.
- This polyhedral surface is compact. Therefore, the spacetime with extreme BTZ-line is *Cauchy-compact* when C/Γ is merely *Cauchy-complete*.

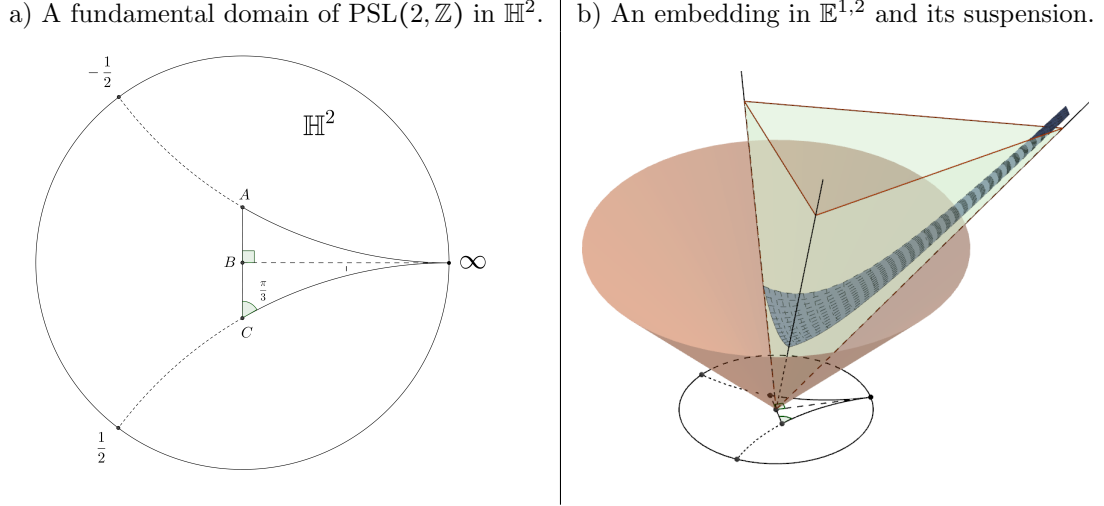


Figure 1: Fundamental domain of the modular group and its suspension.

The fundamental domain of the modular group is represented on the left in the Poincaré disc. The triangles $[AB\infty]$ and $[CB\infty]$ are symmetric with respect to the line $(B\infty)$. The modular group sends the edge $[BC]$ on the edge $[BA]$ via an elliptic isometry of angle π . It sends the edge $[A\infty]$ to $[C\infty]$ via a parabolic isometry of center ∞ . On the right is depicted the natural embedding of this fundamental domain in Minkowski space in deep blue. The light blue cone of triangular basis is its suspension. The null cone of Minkowski space is in red. The stereographic projection of the Poincaré disc is depicted on the horizontal plane $\{t = 0\}$.

This paper is devoted to the description of the process by which BTZ-lines are added and how it interacts with global properties of the spacetime : *global hyperbolicity*, *Cauchy-completeness* and *Cauchy-maximality*. Since a general theory of such singular spacetimes is lacking, part of the paper is devoted to background properties. A following paper will be devoted to the construction of singular Euclidean surfaces in singular spacetimes as well as a correspondance between hyperbolic, Minkowskian and Euclidean objects. Some compactification properties will also be dealt with.

1.2 Structure of the paper and goals

The paper gives the definition of *singular spacetimes* as well as *Cauchy-something* properties and develop some basic properties in section 1.5. Its primary objectives are the following

- I. Define a notion of *BTZ-extension* and prove a maximal BTZ-extension existence and uniqueness theorem. This is Theorem II in Section 3.

- II. Prove that Cauchy-completeness and Cauchy-maximality are compatible with BTZ-extensions. This is Theorem III in section 4.

Some secondary objectives are needed both to complete the picture and to the proofs of the main theorems.

- i. Prove local rigidity property which is an equivalent of local unicity of solution of Einstein equations in our context. This ensure we have a maximal Cauchy-extension existence and uniqueness theorem, much alike the one of Choquet-Bruhat-Geroch, stated in Section 2.2. The local rigidity is done in Section 1.4.3.
- ii. Prove the existence of a smooth Cauchy-surface in a globally hyperbolic singular spacetime. Theorem I proves it in Section 2.3.
- iii. Show that in a Cauchy-maximal spacetime, BTZ-singular lines are complete in the future and posess standard neighborhoods. A proof is given in Section 2.2

1.3 Global properties of regular spacetimes

1.3.1 (G, X) -structures

(G, X) -structures are used in the preliminary of the present work and may need some reminders. Let X be a topological space and $G \subset \text{Homeo}(X)$ be a group of homeomorphism. The couple (G, X) is an analytical structure if two elements of G agreeing on a non trivial open subset of X are equal.

Given (G, X) an analytical structure and M a Hausdorff topological space, a (G, X) -structure on M is the data of an atlas $(\mathcal{U}_i, \varphi_i)_{i \in I}$ where $\varphi_i : \mathcal{U}_i \rightarrow \mathcal{V}_i \subset X$ are homeomorphisms such that for every $i, j \in I$, there exists an element $g \in G$ agreeing with $\varphi_j \circ \varphi_i^{-1}$ on $\mathcal{V}_i \cap \varphi_i(\mathcal{U}_j \cap \mathcal{U}_i)$. A manifold together with a (G, X) -structure is a (G, X) -manifold.

The morphisms $M \rightarrow M'$ of (G, X) -manifolds are the functions $f : M \rightarrow M'$ such that for all couples of charts (\mathcal{U}, φ) and (\mathcal{U}', φ') of M and M' respectively, $\varphi' \circ f \circ \varphi^{-1} : \varphi(\mathcal{U} \cap f^{-1}(\mathcal{U}')) \rightarrow \varphi'(\mathcal{U}')$ is the restriction of an element of G . Given a local homeomorphism $f : M \rightarrow N$ between differentiable manifolds, for every (G, X) -structure on N , there exists a unique (G, X) -structure on M such that f is a (G, X) -morphism.

Writing \widetilde{M} the universal covering of a manifold M and $\pi_1(M)$ its fundamental group, there exists a unique (G, X) -structure on \widetilde{M} such that the projection $\pi : \widetilde{M} \rightarrow M$ is a (G, X) -morphism.

Proposition 1.1 (Fundamental property of (G, X) -structures). *Let M be a (G, X) -manifold. There exists a map $\mathcal{D} : \widetilde{M} \rightarrow X$ called the developping map, unique up to composition by an element of G ; and a morphism $\rho : \pi_1(M) \rightarrow G$, unique up to conjugation by an element of G such that \mathcal{D} is a ρ -equivariant (G, X) -morphism.*

Actually, the analyticity of the (G, X) -structure ensure that every (G, X) -morphism $\widetilde{M} \rightarrow X$ is a developping map.

1.3.2 Minkowski space

The only analytical structure we shall deal with is minkowskian.

Definition 1.2 (Minkowski space). *Let $\mathbb{E}^{1,2} = (\mathbb{R}^3, q)$ be the Minkowski space of dimension 3 where q is the bilinear form $-dt^2 + dx^2 + dy^2$ and t, x, y denote respectively the cartesian coordinates of \mathbb{R}^3 .*

A non zero vector $u \in \mathbb{E}^{1,2} \setminus \{0\}$ is spacelike, lightlike or timelike whether $q(u, u)$ is positive, zero or negative. A vector is causal if it is timelike or lightlike. The set of non zero causal vectors is the union of two convex cones, the one in which t is positive is the future causal cone and the other is the past causal cone.

A continuous piecewise differentiable curve in $\mathbb{E}^{1,2}$ is future causal (resp. chronological) if at all points its tangent vectors are future causal (resp. timelike). The causal (resp. chronological) future of a point $p \in \mathbb{E}^{1,2}$ is the set of points q such that there exists a future causal (resp. chronological) curve from p to q ; it is written $J^+(p)$ (resp. $I^+(p)$).

Consider a point $p \in \mathbb{E}^{1,2}$, we have

- $I^+(p) = \{q \in \mathbb{E}^{1,2} \mid q - p \text{ future timelike}\}$
- $J^+(p) = \{q \in \mathbb{E}^{1,2} \mid q - p \text{ future causal or zero}\}$

The causality defines two order relations on $\mathbb{E}^{1,2}$, the causal relation $<$ and the chronological relation \ll . More precisely, $x < y$ iff $y \in J^+(x) \setminus \{x\}$ and $x \ll y$ iff $y \in I^+(x)$. One can then give the most general definition of causal curve. A causal (resp. chronological) curve is a continuous curve in $\mathbb{E}^{1,2}$ increasing for the causal (resp. chronological) order. A causal (resp. chronological) curve is *inextendible* if every causal (resp. chronological) curve containing it is equal. The causal order relation is often called a *causal orientation*.

Proposition 1.3. *The group $\text{Isom}(\mathbb{E}^{1,2})$ of affine isometries of $\mathbb{E}^{1,2}$ preserving the orientation and preserving the causal order is the identity component of the group of affine isometries of $\mathbb{E}^{1,2}$. Its linear part $\text{SO}_0(1, 2)$ is the identity component of $\text{SO}(1, 2)$.*

A linear isometry either is the identity or possesses exactly one fixed direction. It is elliptic (resp. parabolic, resp. hyperbolic) if its line of fixed points is timelike (resp. lightlike, resp. spacelike). Any $(\text{Isom}(\mathbb{E}^{1,2}), \mathbb{E}^{1,2})$ -manifold, is naturally causally oriented.

Since there are no ambiguity on the group, we will refer to $\mathbb{E}^{1,2}$ -manifold

1.3.3 Globally hyperbolic regular spacetimes

A characterisation of $\mathbb{E}^{1,2}$ -manifolds, to be reasonable, needs some assumptions.

Definition 1.4. *A subset $P \subset M$ of a spacetime M is acausal if any causal curve intersects P at most once.*

Definition 1.5 (Globally hyperbolic $\mathbb{E}^{1,2}$ -structure). *A $\mathbb{E}^{1,2}$ -manifold M is globally hyperbolic if there exists a topological surface Σ in M such that every inextendible causal curve in M intersects Σ exactly once. In particular Σ is acausal. Such a surface is called a Cauchy-surface.*

Definition 1.6 (Cauchy-embedding). *A Cauchy-embedding $f : M \rightarrow N$ between two globally hyperbolic manifolds is an isometric embedding sending a Cauchy-surface (hence every) on a Cauchy-surface.*

We say that N is a Cauchy-extension of M .

A piecewise smooth surface is called *spacelike* if every tangent vector is spacelike. Such a surface is endowed with a metric space structure induced by the ambient $\mathbb{E}^{1,2}$ -structure. If this metric space is metrically complete, the surface is said complete.

Definition 1.7. *A spacetime admitting a metrically complete piecewise smooth and spacelike Cauchy-surface is called Cauchy-complete.*

There is a confusion not to make between Cauchy-complete in this meaning and "metrically complete" which is sometimes referred to by "Cauchy complete" : here, the spacetime is not even a metric space.

Geroch and Choquet-Bruhat [CBG69] proved the existence and uniqueness of the maximal Cauchy-extension of globally hyperbolic Lorentz manifolds satisfying certain Einstein equations (see [Rin09] for a more modern approach). Our special case corresponds to vacuum solutions of Einstein equations. There thus exists a unique maximal Cauchy-extension of a given spacetime.

Mess [Mes07] and then Bonsante, Benedetti, Barbot and others [Bar05], [ABB⁺07], constructed a characterisation of maximal Cauchy-complete globally hyperbolic $\mathbb{E}^{1,n}$ -manifolds for all $n \in \mathbb{N}^*$. This characterisation is based on the holonomy. We are only concerned in the $n = 2$ case.

1.4 Massive particle and BTZ white-hole

1.4.1 Definition and causality

Lorentzian analogue in dimension 3 of conical singularities have been classified in [BBS11]. We are only interested in two specific types we describe below : massive particles and BTZ lines. Massive particles are the most direct Lorentzian analogues of conical singularities. A Euclidean conical singularity can be constructed by quotienting the Euclidean plane by a finite rotation group. The conical angle is then $2\pi/k$ for some $k \in \mathbb{N}^*$. The same way, one can construct examples of massive particles by quotienting $\mathbb{E}^{1,2}$ by some finite group of elliptic isometries.

The general definitions are as follow. Take the universal covering of the complement of a point in the Euclidean plane. It is isometric to $\mathbb{E}_\infty^2 := (\mathbb{R}_+^* \times \mathbb{R}, dr^2 + r^2 d\theta^2)$. The translation $(r, \theta) \mapsto (r, \theta + \theta_0)$ are isometries, one can then quotient out \mathbb{E}_∞^2 by some discrete translation group $\theta_0 \mathbb{Z}$. The quotient is an annulus $\mathbb{R}_+^* \times \mathbb{R}/2\pi\mathbb{Z}$ with the metric $dr^2 + \theta_0 r^2 d\theta^2$ which can be completed by adding one point. The completion is then homeomorphic to \mathbb{R}^2 but the total angle around the origin is θ_0 instead of 2π . Define the model of a massive particle of angle α by the product of a conical singularity of conical angle α by $(\mathbb{R}, -dt^2)$.

Definition 1.8 (Conical singularity). *Let $\alpha \in \mathbb{R}_+^*$. The singular plane of conical angle α , written \mathbb{E}_α^2 , is \mathbb{R}^2 equipped with the metric expressed in polar coordinates*

$$dr^2 + \frac{\alpha}{2\pi} r^2 d\theta^2.$$

The metric is well defined and flat everywhere but at 0 which is a singular point. The space can be seen as the metric completion of the complement of the singular point. The name comes from the fact the metric of a cone in \mathbb{E}^3 can be written this way in a suitable coordinate system. While Euclidean cones have a conical angle less than 2π , a spacelike revolution cone of timelike axis in Minkowski space is isometric to \mathbb{E}_α^2 with α greater than 2π . We insist on the fact that the parameter α is an arbitrary positive real number.

Definition 1.9 (Massive particles model spaces). *Let α be a positive real number. We define :*

$$\mathbb{E}_\alpha^{1,2} := (\mathbb{R} \times \mathbb{E}_\alpha^2, ds^2) \text{ with } ds^2 = -dt^2 + dr^2 + \frac{\alpha}{2\pi} r^2 d\theta^2$$

where t is the first coordinate of the product and (r, θ) the polar coordinates of \mathbb{R}^2 (in particular $\theta \in \mathbb{R}/2\pi\mathbb{Z}$).

The complement of the singular line $\text{Sing}(\mathbb{E}_\alpha^{1,2}) := \{r = 0\}$ is a spacetime called the regular locus and denoted by $\text{Reg}(\mathbb{E}_\alpha^{1,2})$. For $p \in \text{Sing}(\mathbb{E}_\alpha^{1,2})$, we write $]p, +\infty[$ (resp. $[p, +\infty[$) for the open (resp. closed) future singular ray from p . We will also use analogue notation of the past singular ray from p .

Start from a massive particle model space of angle $\alpha \leq 2\pi$, write $\alpha = \frac{2\pi}{\cosh(\beta)}$ and use the coordinates given in definition 1.9. Consider the following change of coordinate.

$$\begin{cases} \tau &= t \cosh(\beta) - r \sinh(\beta) \\ \mathfrak{r} &= \frac{r}{\cosh(\beta)} \\ \theta' &= \theta \end{cases}$$

In the new coordinates, writing $\omega = \tanh(\beta)$, the metric is

$$ds_\omega^2 = \mathfrak{r}^2 d\theta^2 + d\mathfrak{r}^2 - (1 - \omega^2) d\tau^2 - 2\omega d\mathfrak{r} d\tau.$$

Varying ω in $] -1, 1[$, we obtain a continuous 1-parameter family of metrics on \mathbb{R}^3 which parametrises all massive particles of angle less than 2π . The metrics have limits when ω tends toward $\omega = -1$ or $\omega = 1$. The limit metric is non-degenerated, Lorentzian and flat everywhere but on the singular line $\mathfrak{r} = r = 0$. Again the surfaces $\tau = Cte$ are non singular despite the ambient space is. Since the coordinate system of massive particles will not play an important role hereafter, with a slight abuse of notation, we use r instead of \mathfrak{r} coordinate.

Definition 1.10 (BTZ white-hole model space). *The BTZ white-hole model space, noted $\mathbb{E}_0^{1,2}$, is \mathbb{R}^3 equipped with the metric*

$$ds^2 = -2d\tau dr + dr^2 + r^2 d\theta^2$$

where (τ, r, θ) are the cylindrical coordinates of \mathbb{R}^3 . The singular line $\text{Sing}(\mathbb{E}_0^{1,2}) := \{r = 0\}$ is the BTZ line and its complement is the regular locus $\text{Reg}(\mathbb{E}_0^{1,2})$ of $\mathbb{E}_0^{1,2}$. For $p \in \text{Sing}(\mathbb{E}_0^{1,2})$, we write $]p, +\infty[$ (resp. $[p, +\infty[$) for the open (resp. closed) future singular ray from p . We will also use analogue notation of the past singular ray from p .

Remark 1.11 (View points on the Singular line).

- For $\alpha \in \mathbb{R}_+$, notice that the surfaces $\{\tau = \tau_0\}$ are isometric to the Euclidean plane but are not totally geodesic. These surfaces give a foliation of $\mathbb{E}_\alpha^{1,2}$ by surfaces isometric to \mathbb{E}^2 , which is in particular non-singular.
- The ambient Lorentzian space is singular since the metric 2-tensor of $\text{Reg}(\mathbb{E}_\alpha^{1,2})$ does not extend continuously to $\mathbb{E}_\alpha^{1,2}$. Though, as long as $c \in \mathcal{C}_{pw}^1$ then $s \mapsto ds^2(c'(s))$ is piecewise continuous. It can thus be integrated along a curve.

Let $\alpha \in \mathbb{R}_+$. The causal curves are well defined on the regular locus of $\mathbb{E}_\alpha^{1,2}$. The singular line is itself timelike if $\alpha > 0$ and lightlike if $\alpha = 0$. We have to define an orientation on the singular line to define a time orientation on the whole $\mathbb{E}_\alpha^{1,2}$. All the causal curves in $\text{Reg}(\mathbb{E}_\alpha^{1,2})$ share the property that the τ coordinate is monotonic, then we orientate $\text{Sing}(\mathbb{E}_\alpha^{1,2})$ as follows. Curves can be decomposed into a union of pieces of $\text{Sing}(\mathbb{E}_\alpha^{1,2})$ and of curves in the regular locus potentially with ending points on the singular line. Such a curve is causal (resp. chronological) if each part is causal and if the τ coordinate is increasing. The causal future of a point p , noted $J^+(p)$ is then defined as the set of points q such that there exists a future causal curve from p to q . Causal/chronological future/past are defined the same way.

Definition 1.12 (Diamonds). *Let $\alpha \in \mathbb{R}_+$ and let p, q be two points in $\mathbb{E}_\alpha^{1,2}$. Define the closed diamond from p to q :*

$$\overline{\diamond_p^q} = J^+(p) \cap J^-(q)$$

and the open diamond from p to q :

$$\diamond_p^q = \text{Int}(J^+(p) \cap J^-(q))$$

Notice that $\diamond_p^q = I^+(p) \cap I^-(q)$ if $\alpha > 0$. However, if $\alpha = 0$ and p is on the singular line then $I^+(p) = \text{Int}(J^+(p)) \setminus]p, +\infty[$, therefore $I^+(p) \cap I^-(q) = \diamond_p^q \setminus]p, +\infty[$.

The next proposition justifies the name of BTZ white-hole.

Lemma 1.13. *Let $c = (\tau, r, \theta) \in \mathcal{C}^0([0, 1[, \mathbb{E}_0^{1,2})$.*

- (i) *If c is future causal (resp. timelike), then r is increasing (resp. strictly increasing).*
- (ii) *If c is causal future, then c can be decomposed uniquely into*

$$c = \Delta \cup c^0$$

where $c^0 = c \cap \text{Reg}(\mathbb{E}_0^{1,2})$ and $\Delta \subset \text{Sing}(\mathbb{E}_0^{1,2})$ are both connected (possibly empty). Furthermore, Δ lies in the past of c^0 .

Remark that we chose the limit $\omega \rightarrow 1$ to define BTZ white-hole. The limit $\omega \rightarrow -1$ is also meaningful. Its regular part is isometric to $\text{Reg}(\mathbb{E}_0^{1,2})$ as a Lorentzian manifold but the time orientation is reversed. This limit is called BTZ black-hole. We will not make use of them even though one could extend the results presented here to include BTZ black-holes.

Useful neighborhoods of singular points in $\mathbb{E}_\alpha^{1,2}$ are as follows. Take some $\alpha \in \mathbb{R}_+$ and consider the cylindrical coordinates used in the definition of $\mathbb{E}_\alpha^{1,2}$. A tube or radius R is a set of the form $\{r < R\}$, a compact slice of tube is then of the form $\{r \leq R, a \leq \tau \leq b\}$ for $\alpha = 0$ or $\{r \leq R, a \leq t \leq b\}$ for $\alpha > 0$. The abuse of notation between r and τ may induce an imprecision on the radius which may be R or $R/\cosh(\beta)$. However, the actual value of R being non relevant, this imprecision is harmless. More generally an open tubular neighborhood is of the form $\{r < f(\tau), a < \tau < b\}$ where a and b may be infinite.

1.4.2 Universal covering and developping map

Let α be a non-negative real number.

If $\alpha > 0$, the universal covering of $\text{Reg}(\mathbb{E}_\alpha^{1,2})$ can be naturally identified with

$$(\mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}, -dt^2 + dr^2 + r^2 d\theta^2).$$

Definition 1.14. *Define*

$$\mathbb{E}_\infty^{1,2} := (\mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}, -dt^2 + dr^2 + r^2 d\theta^2).$$

Let Δ be the vertical timelike line through the origin in $\mathbb{E}^{1,2}$, the group of isometries of $\mathbb{E}^{1,2}$ which sends Δ to itself is isomorphic to $\text{SO}(2) \times \mathbb{R}$. The $\text{SO}(2)$ factor corresponds to the set of linear isometries of axis Δ and \mathbb{R} to the translations along Δ . The group of isometries of $\mathbb{E}_\infty^{1,2}$ is then the universal covering of $\text{SO}(2) \times \mathbb{R}$, namely $\text{Isom}(\mathbb{E}_\infty^{1,2}) \simeq \widetilde{\text{SO}}(2) \times \mathbb{R}$. The regular part $\text{Reg}(\mathbb{E}_\alpha^{1,2})$ is then the quotient of $\mathbb{E}_\infty^{1,2}$ by the group of isometries generated by $(\tau, r, \theta) \mapsto (\tau, r, \theta + \alpha)$. We will simply write $\alpha\mathbb{Z}$ for this group. There is a natural choice of developping map \mathcal{D} for $\mathbb{E}_\infty^{1,2}$: the projection onto $\mathbb{E}_\infty^{1,2}/2\pi\mathbb{Z}$. Indeed, $\mathbb{E}_\infty^{1,2}/2\pi\mathbb{Z}$ can be identified with the complement of Δ in $\mathbb{E}^{1,2}$. In addition, \mathcal{D} is ρ -equivariant with respect to the actions of $\text{Isom}(\mathbb{E}_\infty^{1,2})$ and $\text{Isom}(\mathbb{E}^{1,2})$ where ρ is the projection onto $\text{Isom}(\mathbb{E}_\infty^{1,2})/2\pi\mathbb{Z} \subset \text{Isom}(\mathbb{E}^{1,2})$. The image of ρ is then the group of rotation-translation around the line Δ with translation parallel to Δ . This couple (\mathcal{D}, ρ) induces a developping map and an holonomy choice for the regular part of $\mathbb{E}_\alpha^{1,2}$ which is $(\mathcal{D}, \rho|_{\alpha\mathbb{Z}})$. We get common constructions, developping map and holonomy for every $\text{Reg}(\mathbb{E}_\alpha^{1,2})$ simultaneously.

Assume now $\alpha = 0$ and let Δ be a lightlike line through the origin in $\mathbb{E}^{1,2}$. Notice that there exists a unique plane of perpendicular to Δ containing Δ since the direction of Δ is lightlike. Let

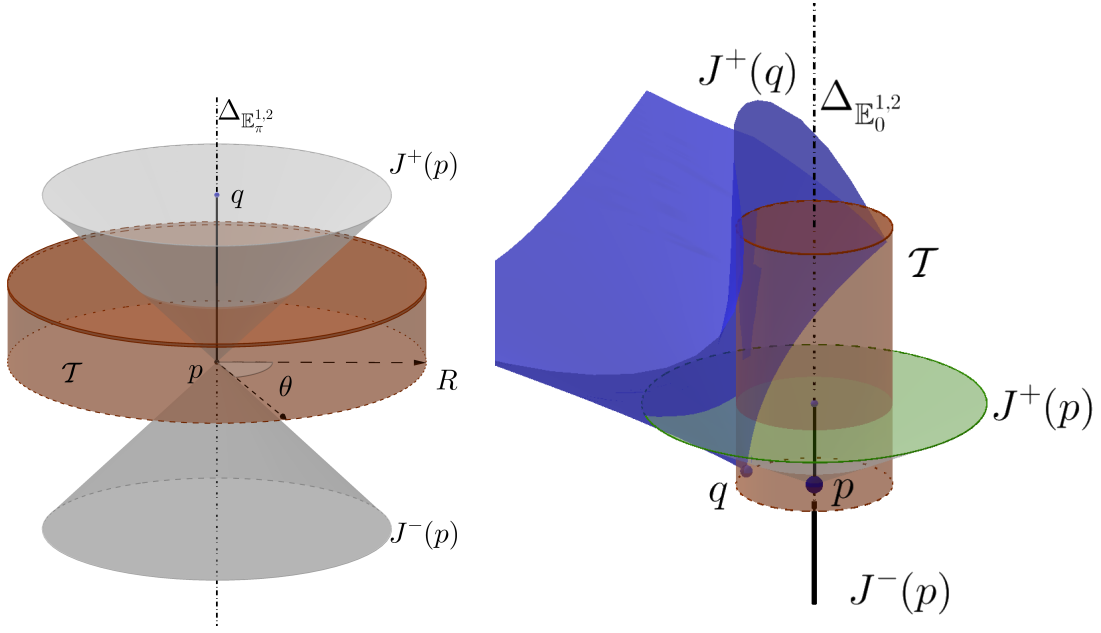


Figure 2: Causal cones in model spaces.

On the left is represented the model space $\mathbb{E}_\pi^{1,2}$. The vertical dotted line is the singular line $\Delta_{\mathbb{E}_\pi^{1,2}}$. On this line are represented two singular points p and q together with causal future and causal past of p in grey. The segment $[p, q]$ is outlined. A tube \mathcal{T} of radius R is represented in brown. The angular coordinate is represented by θ . On the right is represented the model space $\mathbb{E}_0^{1,2}$ with the vertical dotted line as the singular line $\text{Sing}(\mathbb{E}_0^{1,2})$. A singular point p and a regular point q are represented with their causal future. The causal future of p is in green. We have depicted the tube \mathcal{T} containing q in his boundary. The blue surface is the union of future lightlike geodesics starting from q . It does not enter the tube \mathcal{T} . The causal past of p is the black ray below p . The other part of the singular line is the dotted ray above p ; it is the complement of $I^+(p)$ in the interior of $J^+(p)$.

Δ^\perp be the unique plane containing Δ and perpendicular to Δ , we have $I^+(\Delta) = I^+(\Delta^\perp)$. The causal future of Δ is $J^+(\Delta) = I^+(\Delta) \cup \Delta$. The isometries of $\mathbb{E}^{1,2}$ fixing Δ pointwise are parabolic isometries with a translation part in the direction of Δ . The universal covering $\widetilde{\text{Reg}}(\mathbb{E}_0^{1,2})$ can be identified with $\mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}$ endowed with the metric $-2d\tau dr + dr^2 + r^2 d\theta^2$.

Proposition 1.15. *Let $\widetilde{\text{Reg}}(\mathbb{E}_0^{1,2})$ be the universal covering of the regular part of $\mathbb{E}_0^{1,2}$.*

- *The developping map $\mathcal{D} : \widetilde{\text{Reg}}(\mathbb{E}_0^{1,2}) \rightarrow \mathbb{E}^{1,2}$ is injective;*
- *the holonomy sends the translation $(t, r, \theta) \mapsto (t, r, \theta + 2\pi)$ to some parabolic isometry γ which pointwise fixes a lightlike line Δ ;*
- *the image of \mathcal{D} is the chronological future of Δ .*

Proof. Parametrize $\widetilde{\text{Reg}}(\mathbb{E}_0^{1,2})$ by $(\mathbb{R} \times \mathbb{R}_+^* \times \mathbb{R}, -2d\tau dr + dr^2 + r^2 d\theta^2)$. The fundamental group of $\text{Reg}(\mathbb{E}_0^{1,2})$ is generated by the translation $g : (\tau, r, \theta) \mapsto (\tau, r, \theta + 2\pi)$. We use the cartesian coordinates of $\mathbb{E}^{1,2}$ in which the metric is $-dt^2 + dx^2 + dy^2$.

Let $\Delta = \mathbb{R} \cdot (1, 1, 0)$, let γ be the linear parabolic isometry fixing Δ and sending $(0, 0, 1)$ on $(1, 1, 1)$. Then define

$$\mathcal{D}: \begin{cases} \widetilde{\text{Reg}}(\mathbb{E}_0^{1,2}) & \longrightarrow \mathbb{E}^{1,2} \\ \left(\tau, r, \frac{\theta}{2\pi}\right) & \longmapsto \begin{pmatrix} t \\ x \\ y \end{pmatrix} = \begin{pmatrix} \tau + \frac{1}{2}r\theta^2 \\ \tau + \frac{1}{2}r\theta^2 - r \\ -r\theta \end{pmatrix} \end{cases}.$$

A direct computation shows that \mathcal{D} is injective of image $I^+(\Delta)$ and one can see that

- (i) \mathcal{D} is a $\mathbb{E}^{1,2}$ -morphism ;
- (ii) $\mathcal{D}(g \cdot (\tau, r, \theta)) = \gamma \mathcal{D}(\tau, r, \theta)$.

From (i), \mathcal{D} is a developping map and from (ii) the associated holonomy representation sends the translation g to γ . □

Corollary 1.16. *We obtain a homeomorphism $\overline{\mathcal{D}}: \text{Reg}(\mathbb{E}_0^{1,2}) \rightarrow I^+(\Delta)/\langle \gamma \rangle$*

Remark 1.17. *Let \mathcal{D} be a developping map of $\text{Reg}(\mathbb{E}_0^{1,2})$, let γ be a generator of the image of the holonomy associated to \mathcal{D} . Let $\lambda \in \mathbb{R}_+^*$ and let h be a linear hyperbolic isometry of $\mathbb{E}^{1,2}$ which eigenspace associated to λ is the line of fixed points of γ . The hyperbolic isometry h defines an isometry $\text{Reg}(\mathbb{E}_0^{1,2}) \rightarrow \text{Reg}(\mathbb{E}_0^{1,2})$.*

The pullback metric by h is

$$ds_\lambda^2 = -2d\tau dr + dr^2 + \lambda^2 r^2 d\theta^2.$$

Therefore, the above metric on \mathbb{R}^3 is isometric to $\mathbb{E}_0^{1,2}$ for every $\lambda > 0$.

Remark 1.18. *This allows to generalize remark 1.11 above. For all $\lambda > 0$, the coordinates of $\mathbb{E}_0^{1,2}$ above induce a foliation $\{\tau = \tau_0\}$ for $\tau_0 \in \mathbb{R}$. Each leaf is isometric to \mathbb{E}_λ^2 .*

1.4.3 Rigidity of morphisms between model spaces

The next proposition is a rigidity property. A relatively compact subset of $\mathbb{E}^{1,2}$ embeds in every $\mathbb{E}_\alpha^{1,2}$, however we prove below that the regular part of a neighborhood of a singular point in $\mathbb{E}_\alpha^{1,2}$ ($\alpha \neq 2\pi$) cannot be embedded in any other $\mathbb{E}_\beta^{1,2}$. Furthermore, the embedding has to be the restriction of a global isometry of $\mathbb{E}_\alpha^{1,2}$. This proposition is central to the definition of singular spacetime.

Proposition 1.19. *Let $\alpha, \beta \in \mathbb{R}_+$ with $\alpha \neq 2\pi$, and let \mathcal{U} be an open connected subset of $\mathbb{E}_\alpha^{1,2}$ containing a singular point and let ϕ be a continuous function $\mathcal{U} \rightarrow \mathbb{E}_\beta^{1,2}$.*

If the restriction of ϕ to the regular part is an injective $\mathbb{E}^{1,2}$ -morphism then $\alpha = \beta$ and ϕ is the restriction of an element of $\text{Isom}(\mathbb{E}_\alpha^{1,2})$.

Proof. One can assume that \mathcal{U} is a compact slice of tube around the singular line without loss of generality. We use the notation introduced in section 1.4.2.

Assume $\alpha \neq \beta$. Lift ϕ to $\tilde{\phi}: \widetilde{\text{Reg}}(\mathcal{U}) \subset \mathbb{E}_\infty^{1,2} \rightarrow \mathbb{E}_\infty^{1,2}$ equivariant with respect to some morphism $\chi: \alpha\mathbb{Z} \rightarrow \beta\mathbb{Z}$. Writing D the natural projection $\mathbb{E}_\infty^{1,2} \rightarrow \mathbb{E}^{1,2}$, $D|_{\widetilde{\text{Reg}}(\mathcal{U})}$ and $D \circ \tilde{\phi}$ are two developping map of $\text{Reg}(\mathcal{U})$ and are thus equal up to composition by some isometry $\gamma \in \text{Isom}(\mathbb{E}^{1,2})$, we will call the former standard and the latter twisted. Their image is a tube

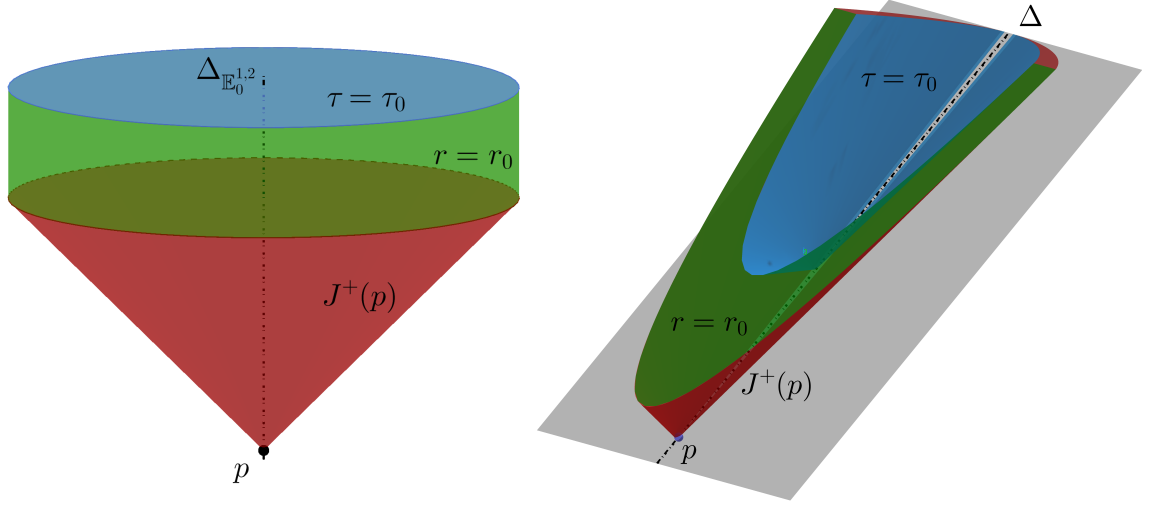


Figure 3: Tubular neighborhood of a BTZ point and its development
On the left, a tubular subset of $\mathbb{E}_0^{1,2}$. On the right its development into $\mathbb{E}^{1,2}$. Colors are associated to remarkable sub-surfaces and their developments.

of respective axis $\Delta = \{r = 0\}$ for the former and $\gamma\Delta$ for the latter. Furthermore, writing ρ the projection of $\text{Isom}(\mathbb{E}_\infty^{1,2}) \rightarrow \text{Isom}(\mathbb{E}^{1,2})$, $\gamma \cdot \rho|_{\alpha\mathbb{Z}} \cdot \gamma^{-1} = (\rho \circ \chi)$.

Assume $\gamma\Delta \neq \Delta$, since the image of D avoids Δ , so does the twisted development of \mathcal{U} . It is then a slice of tube which does not intersect Δ and it is included in some half-space H of $\mathbb{E}^{1,2}$ which support plane contains Δ the vertical axis. Then, by connectedness of \mathcal{U} , the image of $\tilde{\phi}$ is included in some sector $\{\theta_0 \leq \theta \leq \theta_0 + \pi\}$. However, the image should be invariant under $\chi(\alpha\mathbb{Z})$, and the only subgroup of $\beta\mathbb{Z}$ letting such a sector invariant is the trivial one. Consequently $\chi = 0$, thus the lift $\phi : \mathcal{U} \rightarrow \mathbb{E}_\infty^{1,2}$ is well defined and $D \circ \tilde{\phi}$ is injective, then so is $D|_{\widetilde{\text{Reg}}(\mathcal{U})}$. Furthermore, $\rho(\alpha) = 0$, thus $\alpha = 2\pi n$ for some n greater than two. Then D cannot be injective on some loop $\{r = \varepsilon, t = t_0\}$ in \mathcal{U} . Absurd.

Thus $\gamma\Delta = \Delta$, the linear part of γ is an elliptic element of axis Δ and the translation part of γ is in Δ . The isometry γ is then in the image of ρ , one can then assume $\gamma = 1$ by considering $\tilde{\gamma}^{-1}\tilde{\phi}$ instead of $\tilde{\phi}$ with $\rho(\tilde{\gamma}) = \gamma$. In this case, $D \circ \tilde{\phi} := D$ then $\tilde{\phi}$ is a translation of angle $2\pi n$, one can then choose the lift $\tilde{\gamma}$ of γ such that $n = 0$. Consequently, $\tilde{\phi}$ is the restriction to $\widetilde{\text{Reg}}(\mathcal{U})$ of an element of $\text{Isom}(\mathbb{E}_\infty^{1,2})$, ϕ is then a covering and the morphism χ is then the restriction of the multiplication $\text{Isom}(\mathbb{E}_\infty^{1,2}) \xrightarrow{\times n} \text{Isom}(\mathbb{E}_\infty^{1,2})$, then $\alpha\mathbb{Z} = n\beta\mathbb{Z}$ and using again the injectivity of ϕ on a standard loop, one get $\alpha = \beta$.

Assume $\alpha\beta = 0$, one obtain in the same way a morphism φ such that $\rho_\alpha = \rho_\beta \circ \varphi$ induced by a lift $\tilde{\varphi} : \widetilde{\text{Reg}}(\mathbb{E}_\alpha^{1,2}) \rightarrow \widetilde{\text{Reg}}(\mathbb{E}_\beta^{1,2})$. However, $\text{Im}\rho_\alpha$ is generated by an elliptic isometry if $\alpha > 0$ and a parabolic one if $\alpha = 0$, then α cannot be zero if β is not and reciprocally. Then $\alpha = \beta = 0$. One again gets two developments of \mathcal{U} the standard one and the one twisted by some γ , the standard image contains a horocycle around a lightlike line Δ and is invariant exactly under the stabilizer of Δ . The twisted image is then invariant exactly under the stabilizer of $\gamma\Delta$. Therefore, the image of χ is in the intersection of the two and is non trivial. Remark that the only isometries γ such that $\gamma\text{Stab}(\Delta)\gamma^{-1} \cap \text{Stab}(\Delta) \neq \{1\}$ are exactly $\text{Stab}(\Delta)$. Finally, γ stabilizes Δ and one can conclude the same way as before. \square

Remark 1.20. The core argument of the proof above shows that without the hypothesis of injectivity, ϕ shall be induced by a branched covering $\mathbb{E}_\alpha^{1,2} \rightarrow \mathbb{E}_\beta^{1,2}$. Thus $\alpha = n\beta$ for some n and actually knowing that $\alpha = \beta$ gives that ϕ is an isomorphism if $\alpha \neq 0$.

Beware that $\mathbb{E}_0^{1,2}$ is a branched covering of itself since, using cylindrical coordinates $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$, the projection $\mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}/(\frac{2\pi}{p}\mathbb{Z})$, for instance, induces an isometric branched covering $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}/2\pi\mathbb{Z}$, the former with the metric $-d\tau dr + dr^2 + r^2 d\theta^2$ and the latter with the metric $-d\tau dr + dr^2 + \frac{1}{p^2} r^2 d\theta^2$. Both are coordinate systems of $\mathbb{E}_0^{1,2}$ from Remark 1.17. Thus one couldn't get rid of the injectivity condition that easily.

1.5 Singular spacetimes

A singular spacetime is a patchwork of different structures. They can be associated with one another using their regular locus which is a natural $\mathbb{E}^{1,2}$ -manifold. Such a patchwork must be given by an atlas identifying part of M to an open subset of one of the model spaces, the chart must send regular part on regular part whenever they intersect and the regular locus must be endowed with a $\mathbb{E}^{1,2}$ -structure.

Definition 1.21. Let A be a subset of \mathbb{R}_+ . A $\mathbb{E}_A^{1,2}$ -manifold is a second countable Hausdorff topological space M with an atlas $\mathcal{A} = (\mathcal{U}_i, \phi_i)_{i \in I}$ such that

- For every $(\mathcal{U}, \phi) \in \mathcal{A}$, there exists an open set \mathcal{V} of $\mathbb{E}_\alpha^{1,2}$ for some $\alpha \in A$ such that $\phi : \mathcal{U} \rightarrow \mathcal{V}$ is a homeomorphism.
- For all $(\mathcal{U}_1, \phi_1), (\mathcal{U}_2, \phi_2) \in \mathcal{A}$,

$$\phi_2 \circ \phi_1^{-1} [Reg(\phi_1(\mathcal{U}_2 \cap \mathcal{U}_1))] \subset Reg(\phi_2(\mathcal{U}_1 \cap \mathcal{U}_2))$$

and the restriction of $\phi_2 \circ \phi_1^{-1}$ to $Reg(\phi_1(\mathcal{U}_1 \cap \mathcal{U}_2))$ is a $\mathbb{E}^{1,2}$ -morphism.

For $\alpha \in A \setminus \{2\pi\}$, $Sing_\alpha$ denote the subset of M that a chart sends to a singular point of $\mathbb{E}_\alpha^{1,2}$, and $Sing_{2\pi} = \emptyset$.

In the following A is a subset of \mathbb{R}_+ and M is a $\mathbb{E}_A^{1,2}$ -manifold. It is not obvious from the definition that a singular point in M does not admit charts of different types. We need to prove the regular part $Reg(M)$ and the singular parts $Sing_\alpha(M), \alpha \in \mathbb{R}_+$ form a well defined partition of M .

Proposition 1.22. $(Sing_\alpha)_{\alpha \in A}$ is a family of disjoint closed submanifolds of dimension 1.

Proof. Let $\alpha \in \mathbb{R}_+ \setminus \{2\pi\}$ and let $p \in Sing_\alpha$ be a singular point, there exists a chart $\phi : \mathcal{U} \rightarrow \mathcal{V}$ around p such that $\mathcal{V} \subset \mathbb{E}_\alpha$ and such that $\phi(p) \in Sing(\mathbb{E}_\alpha)$. For any other chart $\phi' : \mathcal{U}' \rightarrow \mathcal{V}'$, $\phi' \circ \phi^{-1}(Reg(\mathcal{V}) \cap \phi(\mathcal{U}')) \subset Reg(\mathcal{V}')$ thus $Sing_\alpha \cap \mathcal{U} = \phi^{-1}(Sing(\mathbb{E}_\alpha^{1,2}))$. Since ϕ is a diffeomorphism and $Sing(\mathbb{E}_\alpha^{1,2})$ is a closed 1-dimensional submanifold of $\mathbb{E}_\alpha^{1,2}$, so is $Sing_\alpha \cap \mathcal{U}$. For $p \in Reg(M)$ and \mathcal{U} a chart neighborhood of p , we have $Sing_\alpha \cap \mathcal{U} = \emptyset$. Then $Sing_\alpha$ is a closed 1-dimensional submanifold.

Let $\alpha, \beta \in \mathbb{R}_+^2$ and assume there exists $p \in Sing_\alpha \cap Sing_\beta$. There exists charts $\phi_\alpha : \mathcal{U}_\alpha \rightarrow \mathcal{V}_\alpha$ $\phi_\beta : \mathcal{U}_\beta \rightarrow \mathcal{V}_\beta$ such that $\phi_\alpha(p) \in \Delta_{\mathbb{E}_\alpha^{1,2}}$ and $\phi_\beta(p) \in \Delta_{\mathbb{E}_\beta^{1,2}}$. Then, writing $\mathcal{V}'_\alpha = Reg(\mathcal{V}_\alpha \cap \phi_\alpha(\mathcal{U}_\beta))$ and $\mathcal{V}'_\beta = Reg(\mathcal{V}_\beta \cap \phi_\beta(\mathcal{U}_\alpha))$, $\phi_\beta \circ \phi_\alpha^{-1} : \mathcal{V}'_\alpha \rightarrow \mathcal{V}'_\beta$ is an isomorphism of $\mathbb{E}^{1,2}$ -structures. Since \mathcal{V}'_α is the regular part of an open subset of $\mathbb{E}_\alpha^{1,2}$ containing a singular point, from Proposition 1.19 we deduce that $\alpha = \beta$. □

Definition 1.23 (Morphisms and isomorphisms). *Let M, N be $\mathbb{E}_A^{1,2}$ -manifolds. A continuous map $\phi : M \rightarrow N$ is a $\mathbb{E}_A^{1,2}$ -morphism if $\phi|_{\text{Reg}(M)} : \text{Reg}(M) \rightarrow \text{Reg}(N)$ is a $\mathbb{E}^{1,2}$ -morphism. A morphism ϕ is an isomorphism if it is bijective.*

Consider a $\mathbb{E}_A^{1,2}$ -structure \mathcal{A} on a manifold M and consider thinner atlas \mathcal{A}' . The second atlas defines a second $\mathbb{E}_A^{1,2}$ -structure on M . The identity is an isomorphism between the two $\mathbb{E}_A^{1,2}$ -structures, they are thus identified.

Proposition 1.24. *Let M and N be connected $\mathbb{E}_A^{1,2}$ -manifolds. Let $\phi_1, \phi_2 : M \rightarrow N$ be two $\mathbb{E}_A^{1,2}$ -morphisms. If there exists an open subset $\mathcal{U} \subset M$ such that $\phi_1|_{\mathcal{U}} = \phi_2|_{\mathcal{U}}$ then $\phi_1 = \phi_2$.*

Proof. Since M and N are 3-dimensionnal manifolds and since $\text{Sing}(M)$ and $\text{Sing}(N)$ are embedded 1-dimensional manifolds, $\text{Reg}(M)$ and $\text{Reg}(N)$ are open connected and dense. Since $\text{Reg}(M)$ is a connected $\mathbb{E}^{1,2}$ -structure, $\phi_1|_{\text{Reg}(M)} = \phi_2|_{\text{Reg}(M)}$. By density of $\text{Reg}(M)$ and continuity of ϕ_1 and ϕ_2 , $\phi_1 = \phi_2$. \square

We end this section by an extension to singular manifold of a property we gave for the BTZ model space.

Lemma 1.25. *Let M a $\mathbb{E}_A^{1,2}$ -manifold then*

- *a connected component of $\text{Sing}_0(M)$ is an inextendible causal curve ;*
- *every causal curve c of M decomposes into $c = \Delta \cup c^0$ where $\Delta = c \cap \text{Sing}_0(M)$ and $c^0 = c \setminus \text{Sing}_0(M)$. Furthermore, Δ and c^0 are connected and Δ is in the past of c^0 .*

Proof. A connected component Δ of $\text{Sing}_0(M)$ is a 1-dimensional submanifold, connected and locally causal. Therefore, it is a causal curve. Since it is closed, it is also inextendible.

Assume Δ is non empty and take some $p \in \Delta$. Let $q \in J^-(p)$ and let $c' : [0, 1] \rightarrow J^-(p)$ be a past causal curve such that $c'(0) = p$ and $c'(1) = q$. Then write :

$$I = \{s \in [0, 1] \mid c'([0, s]) \subset \Delta\}.$$

- $0 \in I$ so I is not empty.
- Take $s \in I$, $c'(s)$ is of type $\mathbb{E}_0^{1,2}$ and in a local chart \mathcal{U} , $J_{\mathcal{U}}^-(c'(s))$ is in the singular line around $c'(s)$. Thus for some $\varepsilon > 0$, $c'([s, s + \varepsilon]) \subset \Delta_{c'(s)} = \Delta$. Thus $[0, s + \varepsilon] \subset I$ and I is open.
- Let $s = \sup I$, $c'([s - \varepsilon, s]) \subset S^0$. By closure of S^0 , Δ is closed thus $c'(s) \in \Delta$ and $s \in I$. Then I is closed.

Finally, $I = [0, 1]$ and $q \in \Delta$. We conclude that Δ is connected that there is no point of c^0 in the past of Δ . \square

2 Global hyperbolicity and Cauchy-extensions of singular spacetimes

We remind a Geroch characterisation of globally hyperbolic of regular spacetime and extend it to singular one. We extend the smoothing theorem of Bernal and Sanchez and the the Cauchy-Maximal extension theorem by Geroch and Choquet-Bruhat. We also prove that a BTZ line is complete in the future if the space-time is Cauchy-maximal.

2.1 Global hyperbolicity, Geroch characterisation

Let M be a $\mathbb{E}_A^{1,2}$ -manifold, the causality on M is inherited from the causality and causal orientation of each chart, we can then speak of causal curve, acausal domain, causal/chronological future/past, etc. The chronological past/future are still open (maybe empty) since this property is true in every model spaces. We define global hyperbolicity, give the a Geroch splitting theorem and some properties.

Definition 2.1. *Let $P \subset M$ be a subset of M .*

- *The future Cauchy development of P is the set*

$$D^+(P) = \{x \in M \mid \forall c: [0, +\infty[\rightarrow M \text{ inextendible past causal curve, } c(0) = x \Rightarrow c \cap P \neq \emptyset\}$$

- *The past Cauchy development of P is the set*

$$D^-(P) = \{x \in M \mid \forall c: [0, +\infty[\rightarrow M \text{ inextendible future causal curve, } c(0) = x \Rightarrow c \cap P \neq \emptyset\}$$

- *The Cauchy development of P is the set $D(P) = D^+(P) \cup D^-(P)$*

Definition 2.2 (Cauchy Surface). *A Cauchy-surface in a $\mathbb{E}_A^{1,2}$ -manifold is a \mathcal{C}^0 -surface $\Sigma \subset M$ such that all inextendible causal curves intersects Σ exactly once.*

In particular if Σ is a Cauchy-surface of M then $\mathcal{D}(\Sigma) = M$.

Definition 2.3 (Globally hyperbolic manifold). *If a $\mathbb{E}_A^{1,2}$ -manifold has a Cauchy-surface, it is globally hyperbolic.*

The following theorem gives a fundamental characterisation of globally hyperbolic spacetimes. Neither Geroch nor Bernal and Sanchez have proved this for singular manifolds but the usual arguments apply. The method is to define a time function as a volume function :

$$T(x) = \ln \frac{\mu(I^-(x))}{\mu(I^+(x))}$$

where μ is a finite measure on a spacetime M . Usually, one uses an absolutely continuous measure, however such a measure put a zero weight on the past of a BTZ point. The solution in the presence of BTZ lines is to put weight on the BTZ lines and choosing a measure which is the sum of a 3 dimensional absolutely continuous measure on M and a 1 dimensional absolutely continuous measure on $\text{Sing}_0(M)$. The definition of causal spacetime along with an extensive exposition of the hierarchy of causality properties can be found in [MS08] and a direct exposition of basic properties of such volume functions in [Die88].

Theorem 2.1 ([Ger70],[BS07]). *Let M be a $\mathbb{E}_A^{1,2}$ -manifold, $(i) \Leftrightarrow (ii)$.*

(i) *M is globally hyperbolic.*

(ii) *M is causal and $\forall p, q \in M, \overline{\diamond_p^q}$ is compact.*

Proposition 2.4. *If Σ is a Cauchy-surface of M then there exists a homeomorphism $M \xrightarrow{\phi} \mathbb{R} \times \Sigma$ such that for every $C \in \mathbb{R}$, $\phi^{-1}(\{C\} \times \Sigma)$ is a Cauchy-surface.*

Proof. See [O’N83] □

The topology generated by the open diamonds \diamond_p^q is called the *Alexandrov topology*. In the case of globally hyperbolic spacetimes, the Alexandrov topology coincides with the standard topology on the underlying manifold.

Lemma 2.5. *Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold and let K_1, K_2 be compact subsets. Then*

- $J^+(K_1) \cap J^-(K_2)$ is compact ;
- $J^+(K_1)$ and $J^-(K_2)$ are closed.

Proof. The usual arguments apply since they can be formulated using only the Alexandrov topology, the compactness of closed diamonds and the metrisability of the topology.

Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $J^+(K_1) \cap J^-(K_2)$, there exists sequences $(p_n)_{n \in \mathbb{N}} \in K_1^{\mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}} \in K_2^{\mathbb{N}}$ such that $x_n \in \overline{\diamond_{p_n}^{q_n}}$ for all $n \in \mathbb{N}$. Extracting a subsequence if necessary, one can assume $p_n \xrightarrow{n \rightarrow +\infty} p$ and $q_n \xrightarrow{n \rightarrow +\infty} q$ for some $p \in K_1$ and $q \in K_2$. There exists a neighborhood of p of the form $J^+(p')$ and a neighborhood of q of the form $J^-(q')$ for some p' and q' . The sequences $(p_n)_{n \in \mathbb{N}}$ and $(q_n)_{n \in \mathbb{N}}$ enters respectively $J^+(p')$ and $J^-(q')$, then for n big enough, $x_n \in \overline{\diamond_{p_n}^{q_n}}$. This subset is compact, thus one can extract a subsequence of $(x_n)_{n \in \mathbb{N}}$ converging to some $x_\infty \in \overline{\diamond_{p'}^{q'}}$. Take a sequence $(p'_n)_{n \in \mathbb{N}}$ of such p' 's converging toward p and a sequence $(q'_n)_{n \in \mathbb{N}}$ of such q' 's converging toward q . Since each $J^+(p'_n)$ is a neighborhood of p and each $J^-(q'_n)$ is a neighborhood of q then for all n , $x_k \in \overline{\diamond_{p'_n}^{q'_n}}$ for k big enough. Finally, by compactness of $\overline{\diamond_{p'_n}^{q'_n}}$, the limit

$$x_\infty \in \bigcap_{n \in \mathbb{N}} \overline{\diamond_{p'_n}^{q'_n}} = \overline{\diamond_p^q} \subset J^+(K_1) \cap J^-(K_2)$$

Let $x_n \xrightarrow{n \rightarrow +\infty} x$ be a converging sequence of points of $J^+(K_1)$. There exists a neighborhood of x of the form $J^-(q)$, by global hyperbolicity of M , $J^-(q) \cap J^+(K_1)$ is compact and contains every points of x_n for n big enough. Thus $x \in J^-(q) \cap J^+(K_1)$ and $x \in J^+(K_1)$. We prove the same way that $J^-(K_2)$ is closed. \square

2.2 Cauchy-extension and Cauchy-maximal singular spacetimes

Extensions and maximality of spacetimes are usually defined via Cauchy-embeddings as follows.

Definition 2.6 (Cauchy-embeddings). *Let M_1, M_2 be globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifolds and let $\phi : M_1 \rightarrow M_2$ be a morphism. ϕ is a Cauchy-embedding if it is injective and sends a Cauchy-surface of M_1 on a Cauchy-surface of M_2 .*

The definition can be loosen twice, first by only imposing the existence of a Cauchy-surface of M_1 that ϕ sends to a Cauchy-surface of M_2 , it is an exercise to prove that this implies that every Cauchy-surfaces is sent to a Cauchy-surface. Second, injectivity along a Cauchy-surface implies injectivity of ϕ .

We remind that a spacetime is Cauchy-maximal if every Cauchy-extension is trivial. The proof of the Cauchy-maximal extension theorem given by Choquet-Bruhat and Geroch have been improved by Jan Sbierski in [Sbi15]. This new proof has the advantage of not using Zorn's lemma, it is thus more constructive. The existence and uniqueness of a Maximal Cauchy-extension of a singular spacetime can be proven re-writing the proof given by Sbierski taking some care with the particles. Indeed, it is shown in [BBS11] that collisions of particles can make the uniqueness fail. The rigidity Proposition 1.19 ensures the type of particles is preserved and is an equivalent of local uniqueness of the solution of Einstein's Equation in our context. The proof of separation

given by Sbierski has to be adapted to massive particles of angle greater than 2π to fully work. It can be done without difficulties and the main ideas of the proof are used in section 3.1 for BTZ extensions, so we don't rewrite a proof of the theorem here.

Theorem 2.2 (Cauchy-Maximal extension). *Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold. Then, there exists a maximal Cauchy-extension of M among $\mathbb{E}_A^{1,2}$ -manifold. Furthermore, it is unique up to isomorphism.*

Proof. See [Sbi15]. □

We now prove in a Cauchy-maximal spacetime a BTZ line is complete in the future and there is a standard neighborhood of a future BTZ ray.

Proposition 2.7. *Let M be Cauchy-maximal $\mathbb{E}_A^{1,2}$ -manifold and let p be a BTZ point in $\text{Sing}_0(M)$. Then, the future BTZ ray from p is complete and there exists a future half-tube neighborhood of $[p, +\infty[$ of constant radius.*

Proof. Consider Σ a Cauchy-surface of M . The connected component Δ of p in $\text{Sing}_0(M)$ is an inextendible causal curve thus it intersects the Cauchy-surface Σ exactly once say at $q \in \Sigma \cap \Delta$. There exists a neighborhood $\bar{\mathcal{U}}$ of q isomorphic via some isometry $\phi : \bar{\mathcal{U}} \rightarrow \mathbb{E}_0^{1,2}$ to

$$\{\tau \in [\tau_1, \tau_2], r \leq R\} \subset \mathbb{E}_0^{1,2}$$

for some positive R and reals $\tau_1, \tau_2 \in \mathbb{R}$. Take this neighborhood small enough so that the surface $\{\tau = \tau_2, r < R\}$ is achronal in M . Consider the open tube $\mathcal{T} = \{\tau > \tau_1, r < R\} \subset \mathbb{E}_0^{1,2}$ and $\mathcal{U} = \text{Int}(\bar{\mathcal{U}})$. Define

- $M_0 = M \setminus J^+(\phi^{-1}(\{\tau = \tau_2, r \leq R\}))$;
- $M_2 = (M_0 \amalg \mathcal{T}) / \sim$ with $x \sim y \Leftrightarrow (x \in \mathcal{U}, y \in \mathcal{T} \text{ and } \phi(x) = y)$.

Σ is a Cauchy-surface of M_0 and M is Cauchy-extension of M_0 . In order to prove that M_2 is a $\mathbb{E}_A^{1,2}$ -manifold, we only need to prove it is Hausdorff. Indeed ϕ is an isomorphism thus the union of the atlases of M_0 and \mathcal{T} defines a $\mathbb{E}_A^{1,2}$ -structure on M_2 .

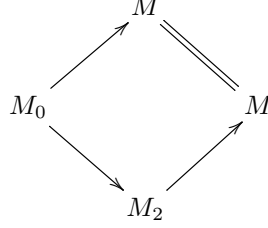
Claim : M_2 is Hausdorff.

Let $x, y \in M_2$, $x \neq y$ and let π be the natural projection $\pi : M_0 \amalg \mathcal{T} \rightarrow M_2$. If $x, y \in \pi(\mathcal{U})$, consider $x_1 = \pi^{-1}(x) \cap \mathcal{U}$, $x_2 = \pi^{-1}(x) \cap \mathcal{T}$, $y_1 = \pi^{-1}(y) \cap \mathcal{U}$, $y_2 = \pi^{-1}(y) \cap \mathcal{T}$. Consider disjoint open neighborhoods \mathcal{V}_{x_1} and \mathcal{V}_{y_1} of x_1 and y_1 . Notice that $\mathcal{V}_x := \pi^{-1}(\pi(\mathcal{V}_{x_1})) = \mathcal{V}_{x_1} \cup \phi(\mathcal{V}_{x_1})$ and that $\mathcal{V}_y := \pi^{-1}(\pi(\mathcal{V}_{y_1})) = \mathcal{V}_{y_1} \cup \phi(\mathcal{V}_{y_1})$. Therefore \mathcal{V}_x and \mathcal{V}_y are open and disjoint neighborhoods of x and y . Notice that $\pi^{-1}(\pi(\mathcal{U})) = \bar{\mathcal{U}} \cup \{\tau \in]\tau_1, \tau_2], r < R\}$. Clearly, if x and y are in $M_2 \setminus \pi(\mathcal{U})$, then they are separated.

Then remains when $x, y \in \partial\pi(\mathcal{U}) = \pi(\partial\mathcal{U}) \cup \pi(\{\tau = \tau_2\})$. Assume $x, y \in \partial\pi(\mathcal{U})$ and consider $x_1 \in \bar{\mathcal{U}}$, $y_1 \in \bar{\mathcal{U}}$ such that $\pi(x_1) = x$ and $\pi(y_1) = y$. Take two disjoint open neighborhoods \mathcal{V}_{x_1} and \mathcal{V}_{y_1} of x_1 and y_1 in M_0 . We have $\pi^{-1}(\pi(\mathcal{V}_{x_1})) = \mathcal{V}_{x_1} \cup \phi(\mathcal{V}_{x_1} \cap \mathcal{U})$ and $\pi^{-1}(\pi(\mathcal{V}_{y_1})) = \mathcal{V}_{y_1} \cup \phi(\mathcal{V}_{y_1} \cap \mathcal{U})$. Then x and y are separated. The same way, we can separate two points $x, y \in \pi(\{\tau = \tau_2, r < R\})$. Assume $x = \pi(x_1)$ with $x_1 \in \partial\bar{\mathcal{U}}$ and $y = \pi(y_1)$ with $y_1 \in \{\tau = \tau_2, r < R\}$. The point x_1 is not in $\phi^{-1}(\{\tau = \tau_2\})$ by definition of M_0 . Therefore, the τ coordinate of $\phi(x_1)$ is less than τ_2 . Take a neighborhood \mathcal{V}_{x_1} of x_1 such that $\phi(\mathcal{V}_{x_1} \cap \mathcal{U}) \subset \{\tau < \tau_2 - \varepsilon\}$ for some $\varepsilon > 0$. Then, take $\mathcal{V}_{y_2} = \{\tau > \tau_2 - \varepsilon, r < R\}$. We get $\pi^{-1}(\pi(\mathcal{V}_{x_1})) = \mathcal{V}_{x_1} \cup \phi(\mathcal{U} \cap \mathcal{V}_{x_1})$ and $\pi^{-1}(\pi(\mathcal{V}_{y_2})) = \mathcal{V}_{y_2} \cup \phi^{-1}(\{\tau \in]\tau_2 - \varepsilon, \tau_2], r < R\})$. Therefore, $\pi(\mathcal{V}_{x_1})$ and $\pi(\mathcal{V}_{y_2})$ are open and disjoint. Finally, M_2 is Hausdorff.

Consider a future inextendible causal curve in M_2 say c and write $\Pi = \{\tau = \tau_2, r < R\} \subset \mathcal{T}$. The curve c can be decomposed into two part : $c_0 = c \cap M_0$ and $c_1 = c \cap J^+(\pi(\{\tau = \tau_2, r < R\}))$. These pieces are connected since Π is achronal in \mathcal{T} and $\phi^{-1}(\Pi)$ is achronal in M . Therefore, c_1 and c_0 are inextendible causal curves if not empty. if c_1 is non empty, then it intersects Π since $D_{\mathcal{T}}^+(\Pi) = \{\tau \geq \tau_2, r < R\}$ and then c_0 is non empty. Therefore, c_0 is always non empty. c_1 does not intersect Σ and c_0 intersects Σ exactly once thus c intersects Σ exactly once.

We obtain the following diagram of extensions by maximality of M



where the arrows are Cauchy-embedding. Therefore, the connected component of p in $\text{Sing}_0(M)$ is complete in the future and has a neighborhood isomorphic to \mathcal{T} . □

2.3 Smoothing Cauchy-surfaces in singular space-times

The question of the existence of a smooth Cauchy-surface of a regular globally hyperbolic manifold has been the object of many endeavours. Seifert [Sei33] was the first one to ask whether the existence of a \mathcal{C}^0 Cauchy-surface was equivalent to the existence of a \mathcal{C}^1 one, he gave a proof which turns out to be wrong. Two recent proofs are considered (so far) to be correct : one of Bernal and Sanchez [BS03] and another by Fathi and Siconolfi [FS12]. We give their result in the case of $\mathbb{E}^{1,2}$ -manifolds.

Theorem 2.3 ([BS03]). *Let M be a globally hyperbolic $\mathbb{E}^{1,2}$ -manifold, then there exists a space-like smooth Cauchy-surface of M .*

We apply their theorem to a globally hyperbolic flat singular spacetime. First we need to define what we mean by spacelike piecewise smooth Cauchy-surfaces. Recall that a smooth surface in $\mathbb{E}^{1,2}$ is *spacelike* if the restriction of the Lorentzian metric to its tangent plane is positive definite.

Definition 2.8. *Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold and let Σ be a Cauchy-surface of M .*

- Σ is smooth (resp. piecewise smooth) if $\Sigma \cap \text{Reg}(M)$ is smooth (resp. piecewise smooth);
- Σ is spacelike (piecewise) smooth if $\Sigma \cap \text{Reg}(M)$ is (piecewise) smooth and spacelike.

Theorem I. *Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold, then there exists a spacelike smooth Cauchy-surface of M .*

Proof. Let Σ_1 be a Cauchy surface of M .

Step 1 Let $(\Delta_i)_{i \in \Lambda}$ be the connected components of $\text{Sing}_0(M)$. Each connected component is an inextendible causal curve intersecting Σ_1 exactly once.

Let $p_i = \Delta_i \cap \Sigma_1$ for $i \in \Lambda_0$. Let $i \in \Lambda_0$, consider $\mathcal{U}_i \simeq \{\tau \in [\tau_i^-, \tau_i^+], r \leq R_i\}$ a tube neighborhood of p_i . Let $\mathbb{D}_i^- = \{\tau = \tau_i^-, r \leq R\}$. The past set $I^-(\Sigma_1 \cap \mathcal{U}_i)$ is an open

neighborhood of the ray $J^-(p_i) \setminus \{p_i\}$. Therefore, noting $q_i = \mathbb{D}_i^- \cap J^-(p_i)$, the past of Σ_1 contains a neighborhood of q_i in \mathbb{D}_i . Reducing R_i if necessary, one can assume $\mathbb{D}_i^- \subset I^-(\Sigma_1)$ and reducing R_i even more we can assume $\mathbb{D}_i^+ := \{\tau = \tau_i^+, r \leq R_i\} \subset J^+(p_i)$. In the same way, we index the connected components of the set of massive particles by Λ_{mass} . We have $\bigcup_{\alpha>0} \text{Sing}_\alpha(M) = \bigcup_{j \in \Lambda_{\text{mass}}} (\Delta_j)$ and $p_j = \Delta_j \cap \Sigma_1$ for $j \in \Lambda_{\text{mass}}$. Since $\Lambda \cup \Lambda_{\text{mass}}$ is enumerable, one can construct $(U_{i_n})_{n \in \mathbb{N}}$ by induction such that for all $n \in \mathbb{N}$, $\mathcal{U}_{i_{n+1}}$ is disjoint from $J^\pm(\mathcal{U}_{i_k})$ for $k \leq n$. Then define

$$N = \left(\bigcup_{i \in \Lambda} J^+(\mathbb{D}_i^+) \cup J^-(\mathbb{D}_i^-) \right) \cup \left(\bigcup_{j \in \Lambda_{\text{mass}}} J^+(p_j) \cup J^-(p_j) \right)$$

and

$$M' = \text{Reg}(M \setminus N).$$

The closed dimonds of M' are compact, thus by Theorem 2.1, M' is a globally hyperbolic $\mathbb{E}^{1,2}$ -manifold. Theorem 2.3 then ensures there exists a smooth Cauchy surface Σ_2 of M' . We need to extend Σ_2 to get a Cauchy-surface of M .

Step 2 We write \mathbb{D}_R the compact disc of radius R in \mathbb{E}^2 and $\mathbb{D}_R^* := \mathbb{D}_R \setminus \{0\}$. Consider a massive particle point p_j for some $j \in \Lambda_{\text{mass}}$ and a tube neighborhood $\mathcal{U} \simeq \{t \in [t^-, t^+], r \leq R\}$ of p_j in M . We may assume the t coordinate of p_j to be 0, $t^- = -t^+$ and $R = t^+$ so that $\{t = t^+\}$ is exactly the basis of the cone $J^+(p_j)$ in \mathcal{U} and $\{t = t^-\}$ is exactly the basis of the cone $J^-(p_j)$ in \mathcal{U} . Consider the projection

$$\pi : \begin{array}{ccc} (\Sigma_2 \cap \mathcal{U}) \cup \{p_j\} & \longrightarrow & \mathbb{D}_R \\ (t, r, \theta) & \longmapsto & (r, \theta) \end{array}$$

where (t, r, θ) are the cylindrical coordinates of \mathcal{U} . The projection π is continuous. Notice that for $r_0 \in]0, R]$ and $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$, the causal curves $\{r = r_0, \theta = \theta_0, t \in]-r_0, r_0[\}$ are inextendible in M' . They thus intersect Σ_2 exactly once and π is thus bijective. Let $(q_n)_{n \in \mathbb{N}}$ be a sequence of points of Σ such that the r coordinates tends to 0. Writing r_n and t_n the r and t coordinates of q_n for $n \in \mathbb{N}$, we have $|t_n| < r_n$ thus $q_n \rightarrow p_j$. Since \mathcal{U} is compact, $\Sigma_2 \cap \mathcal{U} \cup \{r \geq R_1\}$ is compact, it follows that $(\Sigma_2 \cap \mathcal{U}) \cup \{p_j\}$ is compact. Then π is a homeomorphism.

Consider now a BTZ point p_i for some $i \in \Lambda_0$ and a chart neighborhood of p_i as in the first step. Again, the Cauchy-surface Σ_2 is trapped between \mathbb{D}_i^+ and \mathbb{D}_i^- , the projection $\pi : \Sigma_2 \cap \mathcal{U} \rightarrow \mathbb{D}_R^*$ is bijective and open thus a homeomorphism. Write $\pi^{-1} = (\tau, Id)$ and let $(a_n)_{n \in \mathbb{N}}, (b_n)_{n \in \mathbb{N}}$ be two sequences of points of \mathbb{D}_R^* which tend to 0. By compactness, we can assume $(\tau(a_n))_{n \in \mathbb{N}}$ and $(\tau(b_n))_{n \in \mathbb{N}}$ converge to some τ_a and τ_b respectively. If $\tau_a < \tau_b$ then for n big enough $(\tau_a, 0) \in I^-(\pi^{-1}(b_n))$ which is open. Therefore, there exists $n, m \in \mathbb{N}$ such that $\pi^{-1}(a_n) \in I^-(\pi^{-1}(b_m))$. Since Σ_2 is acausal this is absurd and $\tau_a = \tau_b$. Then π^{-1} can be extended to a homeomorphism $\mathbb{D}_R \rightarrow (\Sigma_2 \cap \mathcal{U}) \cup \{q_i\}$ for some $q_i \in \Delta_i$.

Define $\Sigma = \Sigma_2 \cup \{p_j : j \in \Lambda_{\text{mass}}\} \cup \{q_i : i \in \Lambda\} = \bar{\Sigma}_2$, it is a topological surface smooth on the regular part.

Step 3 We need to show Σ_2 is a Cauchy-surface of M . Let c be a future causal inextendible curve in M . Notice that N can be decomposed

$$N = N^+ \cup N^- \quad \text{where} \quad N^\pm = \bigcup_{i \in \Lambda_0} J^\pm(\mathbb{D}_i^\pm) \cup \bigcup_{j \in \Lambda_{\text{mass}}} J^\pm(\{p_j\})$$

is the future complete part (the union of the J^+ parts) and N^- is the past complete part (the union of the J^- parts). A future causal curve entering N^+ cannot leave N^+ and a future causal curve leaving N^- cannot re-enter N^- . Therefore, c is decomposed into three connected pieces $c = c^- \cup c^0 \cup c^+$, the pieces c^-, c^0, c^+ being in $N^-, M \setminus N$ and N^+ respectively.

- If $c^0 = \emptyset$ then $c \subset nN$ thus $c \cap \Sigma_1 \subset N \cap \Sigma = \bigcup_{j \in \Lambda_{\text{mass}}} \{p_j\}$. The curve c can then only intersect Σ at a massive particle point, but such points of Σ are in Σ_1 which is a Cauchy-surface of M . Therefore, c intersects Σ exactly once at p_j for some $j \in \Lambda_{\text{mass}}$.
- If $c^0 \neq \emptyset$, then by Lemma 1.13, c^0 decomposes into a BTZ part Δ and a non BTZ-part c^1 with Δ in the past of c^1 . Then $c = c^- \cup \Delta \cup c^1 \cup c^+$.
 - * If $c^1 \neq \emptyset$, then c^1 is an inextendible causal curve in M' and thus intersect Σ_2 , thus Σ , exactly once. If $q_i \in \Delta$ for some $i \in \Lambda_0$, then $q_i \in J_M^-(\Sigma_2) \setminus \Sigma_2 = I^-(\Sigma_2)$. However, $I^-(\Sigma_2)$ is open and $q_i \in \Sigma_2$, thus $I^-(\Sigma_2) \cap \Sigma_2 \neq \emptyset$ which is absurd since Σ_2 is acausal in M' . Thus $c \cap \Sigma = c^1 \cap \Sigma$ which is a singleton.
 - * If $c^1 = \emptyset$, then Δ is inextendible and thus a connected component of $\text{Sing}_0(M \setminus N)$. Such a connected component contains exactly one of the $(q_i)_{i \in \Lambda}$ thus $c \cap \Sigma = \Delta \cap \Sigma$ is a singleton.

□

Lemma 2.9. *Let Σ a piecewise spacelike Cauchy-surface of M and write $\mathbb{D}_R = \{r \leq R\} \subset \mathbb{E}^2$. For all $p \in \Sigma$, there exists a tube neighborhood $\mathcal{U} \simeq \mathcal{T} \subset \mathbb{E}_\alpha^{1,2}$ of p such that*

- $\mathcal{T} = \{\tau \in [\tau_1, \tau_2], r \leq R\}$ if $\alpha = 0$;
- $\mathcal{T} = \{t \in [t_1, t_2], r \leq R\}$ if $\alpha > 0$;
- $\Sigma \cap \mathcal{U} = \{(f(r, \theta), r, \theta) : (r, \theta) \in \mathbb{D}_R\}$ for some $f : \mathbb{D}_R \rightarrow \mathbb{R}$ which is piecewise smooth on \mathbb{D}_R^* .

Proof. Steps 1 and 2 of the proof of Theorem I give a continuous parametrisation. The parametrisation is piecewise smooth since the projection is along lightlike line or timelike line which are transverse to the spacelike Cauchy-surface. □

The Riemann metric on $\text{Reg}(\Sigma)$ induces a length space structure on $\text{Reg}(\Sigma)$ and a distance function on $\text{Reg}(\Sigma) \times \text{Reg}\Sigma$. In the next proposition, we extend this length space structure on the whole Σ by proving C_{pw}^1 curve to the whole Σ

Proposition 2.10. *Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold and let Σ a piecewise smooth spacelike Cauchy-surface. Then the distance function on $\text{Reg}(\Sigma) \times \text{Reg}\Sigma$ extends continuously to $\Sigma \times \Sigma$.*

Proof. We just have to prove it in the neighborhood of a singular point. There are two cases wether the singular point is BTZ or massive. Let $p \in \Sigma \cap \text{Sing}_0$, consider a local parametrisation given by Lemma 2.9 by a disc of radius $R > 0$ in a compact tube neighborhood \mathcal{U} of p . Take a curve on $\Sigma \cap \mathcal{U}$, $c = (\tau(r), r, \theta_0)$ for $r \in [R, 0]$, it is absolutely continuous on $[R, 0[$. Since Σ is spacelike using the metric of BTZ model space given in Definition 1.10, we have $1 - 2\frac{\partial \tau}{\partial r} \geq 0$

almost everywhere and

$$length(c) = \int_0^R \sqrt{1 - 2 \frac{\partial \tau}{\partial r}} dr \quad (1)$$

$$\leq \int_0^R \sqrt{1 + 1 - 2 \frac{\partial \tau}{\partial r}} dr \quad (2)$$

$$\leq \int_0^R \left(2 - 2 \frac{\partial \tau}{\partial r}\right) dr \quad (3)$$

$$\leq 2R - 2(\tau(R) - \tau(0)) \quad (4)$$

Then the distance induced by the Riemann metric on $Reg(\Sigma)$ extends continuously to p .

Let $p \in \Sigma \cap \text{Sing}_\alpha$ for some $\alpha > 0$, consider a local parametrisation given by Lemma 2.9 by a disc of radius $R > 0$ in a compact tube neighborhood \mathcal{U} of p . Take a curve on $\Sigma \cap \mathcal{U}$, $c = (t(r), r, \theta_0)$ for $r \in [R, 0]$, it is absolutely continuous on $[R, 0[$. Since Σ is spacelike using the metric of massive particle model space given in 1.9, $1 - \left(\frac{\partial t}{\partial r}\right)^2 \geq 0$ and

$$length(c) = \int_0^R \sqrt{1 - \left(\frac{\partial t}{\partial r}\right)^2} dr \quad (5)$$

$$\leq R \quad (6)$$

Then the distance induced by the Riemann metric on $Reg(\Sigma)$ extends continuously to p . \square

Definition 2.11. Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold, M is Cauchy-complete if there exists a piecewise smooth spacelike Cauchy-surface (in the sense of definition 2.8) which is complete as metric space.

Remark 2.12. Fathi and Siconolfi in [FS12] proved a smoothing theorem applicable in a wider context than the one of semi-riemannian manifolds. However their result does not apply naively to our singularities. Consider M a differentiable manifold, their starting point is a continuous cone field i.e. a continuous choice of cones in $T_x M$ for $x \in M$. If one start from a spacetime using our definition, a natural cone field associate to each point x the set of future pointing causal vectors from x . However, as shown on figure 4 this cone field is **discontinuous** at every singular points!

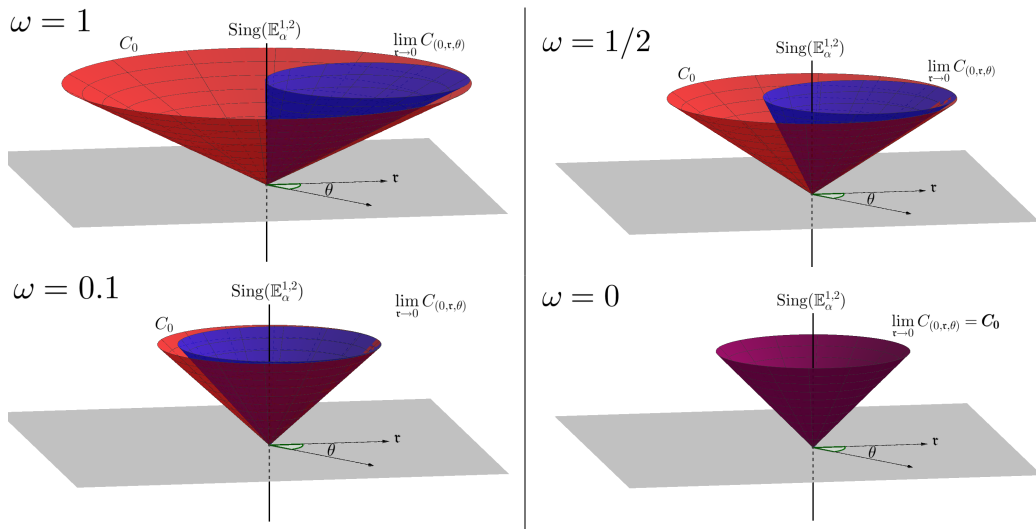


Figure 4: Futur Cones and limit futur cones in the tangent plane of a singular point. We draw cones of future pointing causal vectors in $T_p \mathbb{E}_\alpha^{1,2}$ with $\alpha = 2\pi\sqrt{1-\omega^2}$ and $p \in \text{Sing}(\mathbb{E}_\alpha^{1,2})$ in the (τ, r, θ) coordinates. The red cone C_0 represents the cone of future pointing causal vectors at p and the blue cone represents the radial limit toward p of the cone of future pointing causal vectors. When $\omega = 0$ (i.e. $\alpha = 2\pi$), there the vertical line is not singular anymore and the red and blue cones blend.

It would be possible to construct a continuous cone field which contains the one of future causal vectors. Providing this new cone field is globally hyperbolic (in the sense of Fathi and Siconolfi) one could then apply the smoothing theorem and recover an everywhere smooth Cauchy-surface. This procedure might be slightly simpler and is much stronger since it allows to control both the position and the tangent plane of the Cauchy-surface. We didn't write this here since we also needed Lemma 2.9 and we should have written the first two steps of the presented method anyway. Later on in this paper, we will need to control Cauchy-completeness of Cauchy-surfaces which presents, to our knowledge, the same difficulties using either Fathi-Siconolfi theorem or our method.

Still, it would be nice to have an extended Fathi-Siconolfi theorem which directly applies. This would require to weaken the continuity hypothesis to some semi-continuity hypothesis which seems reasonable considering the methods they used.

3 Catching BTZ-lines : BTZ-extensions

In the example of the modular group presented in the Introduction, we added BTZ-lines to a Cauchy-maximal spacetime. Therefore, the usual Cauchy-maximal extension theorem doesn't catch them. Since we want to add BTZ-lines to some given manifold whenever it is possible, we define BTZ-extensions and prove a corresponding maximal BTZ-extension theorem.

3.1 BTZ-extensions, definition and properties

Consider the regular part of $\mathbb{E}_0^{1,2}$. It is a Cauchy-maximal globally hyperbolic $\mathbb{E}^{1,2}$ -manifold and should be naturally extended into $\mathbb{E}_0^{1,2}$. To get this we need new extensions. Let $A \subset \mathbb{R}_+$ be a

subset containing 0.

Definition 3.1 (BTZ-embedding, BTZ-extension). *Let M_1, M_2 be two globally hyperbolic $\mathbb{E}_A^{1,2}$ manifolds and $\phi: M_1 \rightarrow M_2$ a morphism of $\mathbb{E}_A^{1,2}$ -structure.*

If ϕ is injective and the complement of its image in M_2 is a union (possibly empty) of BTZ lines then ϕ is a BTZ-embedding and M_2 is a BTZ-extension of M_1 .

The following lemmas ensure that two BTZ lines cannot be joined via an extension and that the BTZ-lines cannot be completed in the future via BTZ-extensions.

Lemma 3.2. *Let M_1 and M_2 be two globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifolds, and $M_1 \xrightarrow{\phi} M_2$ a BTZ extension. Let $p, q \in \text{Sing}_0(M_1)$, if p and q are in the same connected component of $\text{Sing}_0(M_2)$ then they are in the same connected component of $\text{Sing}_0(M_1)$.*

Proof. The connected component of p in $\text{Sing}_0(M_2)$ is an inextendible causal curve we note Δ . We may assume $p \in J_{M_2}^-(q)$. Since every point of $[p, q]$ is locally modeled on $\mathbb{E}_0^{1,2}$, we can construct a tube neighborhood of $[p, q]$ of some radius R . Take some regular point q' in the chronological future of q in the tube neighborhood. The diamond $J_{M_1}^-(q') \cap J_{M_1}^+(p)$ is compact, thus $[p, q] \subset \overline{I^+(p) \cap I^-(q')} \subset J_{M_1}^-(q') \cap J_{M_1}^+(p) \subset M_1$. \square

Lemma 3.3. *Let M_1 and M_2 be two globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifolds, and $M_1 \xrightarrow{\phi} M_2$ a BTZ extension. Let $p, q \in \text{Sing}_0(M_2)$.*

If $p \in J_{M_2}^-(q)$ and $p \in M_1$ then $q \in M_1$.

Proof. Take some tube neighborhood \mathcal{T} of radius R of $[p, q]$. Take some point $q' \in (\partial J_{M_2}^+(q)) \cap \mathcal{T}$ then $p \in I_{M_2}^-(q')$. The diamond $\overline{\diamond_p^{q'}}$ in M_1 is compact and its interior is the open diamond $\diamond_p^{q'}$. The latter is relatively compact in M_2 and the former contains its closure in M_2 which contains $[p, q]$. Thus $[p, q]$ is in M_1 . \square

3.2 Maximal BTZ-extension theorem

We can now address the maximal BTZ-extension problem for globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifolds. More precisely, we prove the following theorem.

Theorem II (Maximal BTZ-extension). *Let $A \subset \mathbb{R}_+$, let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold.*

There exists a maximal BTZ-extension \overline{M} of M . Furthermore it is unique up to isometry.

Again, we mean that a spacetime is BTZ-maximal if any BTZ-extension is surjective hence an isomorphism. The proof has similarities with the one of the maximal Cauchy-extension theorem. Let M a spacetime and consider two BTZ-embeddings $f: M_0 \rightarrow M_1$ and $g: M_0 \rightarrow M_2$.

Definition 3.4. *Define $M_1 \wedge M_2$ the union of extensions M of M_0 in M_1 such that there exists a BTZ-embedding $\phi_M: M \rightarrow M_2$ with $\phi_M \circ f = g$.*

Definition 3.5 (Greatest common sub-extension). *Define*

$$\phi: \begin{array}{l|l} M_1 \wedge M_2 & \longrightarrow M_2 \\ x & \longmapsto \phi_M(x) \text{ if } x \in M \end{array}$$

This function ϕ is well defined since each ϕ_M is continuous and $f(M_0)$ is dense in M_1 .

Proposition 3.6. *ϕ is a BTZ-embedding.*

Proof. The image of ϕ contains the image of M_0 thus its complement in M_2 is a subset of $\text{Sing}_0(M_2)$. We must show ϕ is injective.

Let N_1 and N_2 be two subextensions of M_1 together with BTZ-embeddings $\phi_i : N_i \rightarrow M_2$. Let $(x, y) \in N_1 \cup N_2$ be such that $\phi(x) = \phi(y) = p \in M_2$. Notice that $I^+(p) \subset M_2 \setminus \text{Sing}_0(M_2)$ thus

$$\emptyset \neq I^+(p) = \phi_1(I^+(x)) = \phi_2(I^+(y)) \subset g(M_0).$$

Then $I^+(x) = f \circ g^{-1}(I^+(p)) = I^+(y)$ and $x = y$. \square

Definition 3.7 (Least common extension). *Define the least common extension of M_1 and M_2 :*

$$M_1 \vee M_2 = (M_1 \coprod M_2) / (M_1 \wedge M_2)$$

where the quotient is understood identifying $M_1 \wedge M_2$ and $\phi(M_1 \wedge M_2)$. The define the natural projection

$$\pi : M_1 \coprod M_2 \rightarrow M_1 \vee M_2.$$

The following diagram sums-up the situation.

$$\begin{array}{ccccc}
 & & M & \longrightarrow & M_1 \wedge M_2 & \longrightarrow & M_1 & & \\
 & \nearrow f & & & \downarrow \phi_M & & \downarrow \phi & & \\
 M_0 & & & & & & & & \\
 & \searrow g & & & & & & & \\
 & & & & M_2 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & M_1 \vee M_2 & & & &
 \end{array}$$

Notice that $M_1 \vee M_2$ need not be Hausdorff. There could be non separated points, i.e. points p, q such that for every couple of open neighborhoods $(\mathcal{U}, \mathcal{V})$ of p and q , $\mathcal{U} \cap \mathcal{V} \neq \emptyset$.

Definition 3.8. *Define $C = \{p \in M_1 \wedge M_2 \mid \pi(p) \text{ is not separated}\}$*

The following Propositions prove that $(M_1 \wedge M_2) \cup C$ is a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold to which ϕ extends. Thus it is a sub-BTZ-extension common to M_1 and M_2 . It will prove that $C = \emptyset$.

Proposition 3.9. *$(M_1 \wedge M_2) \cup C$ is open and ϕ extends injectively to $(M_1 \wedge M_2) \cup C$.*

Proof. Since $M_1 \wedge M_2$ is a $\mathbb{E}_A^{1,2}$ -manifold we shall only check the existence of a chart around points of C . The set C is in the complement of $M_1 \wedge M_2$ and thus is a subset of $\text{Sing}_0(M_1)$. Let $p \in C$ and $p' \in M_2$ such that $\pi(p)$ and $\pi(p')$ are not separated in $M_1 \vee M_2$. Let $\mathcal{U}_p \xrightarrow{\psi_p} \mathcal{V}_p \subset \mathbb{E}_0^{1,2}$ a chart around p and $\mathcal{U}_{p'} \xrightarrow{\psi_{p'}} \mathcal{V}_{p'} \subset \mathbb{E}_\alpha^{1,2}$ a chart around p' . Since $\pi(p)$ and $\pi(p')$ are not separated, there exists a sequence $(p_n)_{n \in \mathbb{N}} \in \text{Reg}(M_1)^\mathbb{N}$ such that $\lim_{n \rightarrow +\infty} p_n = p$ and $\lim_{n \rightarrow +\infty} \phi(p_n) = p'$. Take such a sequence, notice that for all $n \in \mathbb{N}$, $\phi(I^+(p_n)) = I^+(\phi(p_n))$ and that $I^+(p) \subset \bigcup_{n \in \mathbb{N}} I^+(p_n)$ and $I^+(p') \subset \bigcup_{n \in \mathbb{N}} I^+(\phi(p_n))$. We then get $\phi(I^+(p)) = I^+(p')$. Therefore, taking smaller \mathcal{U}_p and $\mathcal{U}_{p'}$ if necessary, we may assume, \mathcal{U}_p connected and $\phi(I^+(p) \cap \mathcal{U}_p) = I^+(p') \cap \mathcal{U}_{p'}$ then

$$\psi_{p'} \circ \phi \circ \psi_p^{-1} : I^+(\psi_p(p)) \cap \mathcal{V}_p \rightarrow I^+(\psi_{p'}(p')) \cap \mathcal{V}_{p'}$$

is an injective $\mathbb{E}^{1,2}$ -morphism. The future of a point in \mathcal{V}_p is the regular part of a neighborhood of some piece of the singular line in $\mathbb{E}_0^{1,2}$. Thus, by Proposition 1.19, $\alpha = 0$ and $\psi_{p'} \circ \phi \circ \psi_p^{-1}$ is the restriction of an isomorphism of $\mathbb{E}_0^{1,2}$ say γ_p .

Choose such neighborhood \mathcal{U}_p , such $\psi_p, \psi_{p'}$ and such γ_p for all $p \in C$. The subset $(M_1 \wedge M_2) \cup \bigcup_{p \in C} \mathcal{U}_p$ is open thus a $\mathbb{E}_A^{1,2}$ -manifold and the $\mathbb{E}_A^{1,2}$ -morphism

$$\bar{\phi}: \begin{cases} (M_1 \wedge M_2) \cup \bigcup_{p \in C} \mathcal{U}_p & \longrightarrow M_2 \\ x & \longmapsto \begin{cases} \phi(x) & \text{if } x \in M_1 \wedge M_2 \\ \psi_{p'}^{-1} \circ \gamma \circ \psi_p(x) & \text{if } x \in \mathcal{U}_p \end{cases} \end{cases}$$

is then well defined by Lemma 1.24. Notice that for all $p \in C$ and all $q \in \text{Sing}_0(\mathcal{U}_p)$, the points $\pi(q)$ and $\pi \circ \bar{\phi}(q)$ of $M_1 \vee M_2$ are not separated. Therefore, either $q \in C$ or $\pi(q) = \pi \circ \bar{\phi}(q)$ thus

$$(M_1 \wedge M_2) \cup \bigcup_{p \in C} \mathcal{U}_p = (M_1 \wedge M_2) \cup C$$

and thus $(M_1 \wedge M_2) \cup C$ is open.

It remains to show that $\bar{\phi}$ is injective. If $p, q \in (M_1 \wedge M_2) \cup C$ have same image by $\bar{\phi}$ then the image by $\bar{\phi}$ of any neighborhood of p intersects the image of any neighborhood of q . This intersection is open and thus contains regular points. We can construct sequences of regular points $p_n \rightarrow p$ and $q_n \rightarrow q$ such that $\phi(p_n) = \phi(q_n)$. By injectivity of ϕ , $\forall n \in \mathbb{N}, p_n = q_n$ and thus, since M_1 is Hausdorff, $p = q$. □

Proposition 3.10. $(M_1 \wedge M_2) \cup C$ is globally hyperbolic.

Proof. Write $M = (M_1 \wedge M_2) \cup C$ and let $p, q \in M$, we now show that $J_M^-(q) \cap J_M^+(p)$ is compact. We identify M_0 and $f(M_0) \subset M_1$. If $p \notin \text{Sing}_0(M_1)$, $J^+(p) \subset M \setminus \text{Sing}_0(M_1) \subset M_0$ and $J_M^-(q) \cap J_M^+(p) = J_{M_0}^-(q) \cap J_{M_0}^+(p)$ which is compact. Assume now p is of type $\mathbb{E}_0^{1,2}$. Let $(x_n) \in M^{\mathbb{N}}$ be sequence such that $\forall n \in \mathbb{N}, x_n \in J_M^+(p) \cap J_M^-(q)$. By compactness of $J_{M_1}^+(p) \cap J_{M_1}^-(q)$, we can assume (x_n) converges to some $x \in \text{Sing}_0(M_1)$.

Consider some compact tube slice neighborhood \mathcal{T} of $[p, x]$ in M_1 , the subset $M \cap [p, x]$ is open in $[p, x]$ and contains p . Consider $I = \{y \in [p, x] \mid [p, y] \subset M\}$. The set I is connected and open in $[p, x]$. Take an increasing sequence $(y_n)_{n \in \mathbb{N}} \in I^{\mathbb{N}}$, it converges toward some $y_\infty \in [p, x]$. Take some compact diamond neighborhood $\widehat{\diamond}_p^{p'}$ of $]p, y_\infty[$ inside \mathcal{T} . We can take $p' \in \partial J^+(y_\infty)$ so that $p' \in M_0 \cap \mathcal{T}$. The diamond $\phi(\widehat{\diamond}_p^{p'})$ of M_2 is relatively compact thus one can extract a converging subsequence of $\phi(y_n)$ toward some $y'_\infty \in M_2$. Therefore $\pi(y'_\infty)$ and $\pi(y_\infty)$ are not separated and $y_\infty \in M$. Finally, $M \cap [p, x]$ is closed and $I = [p, x]$. Finally, $x \in M$, the sequence $(x_n)_{n \in \mathbb{N}}$ has a converging subsequence in $J_M^+(p) \cap J_M^-(q)$. □

Corollary 3.11. $M_1 \vee M_2$ is Hausdorff.

Proof. $(M_1 \wedge M_2) \cup C$ is a BTZ-extension of M_0 inside M_1 with a BTZ-embedding into M_2 . Therefore it is a subset of $M_1 \wedge M_2$ by maximality of $M_1 \wedge M_2$. Finally, $C = \emptyset$. □

The construction above of a least common extension show that the family of BTZ-extensions of M_0 is a right filtered family and can thus take the direct limit of all such extensions. Consider a family of representants of the isomorphism classes $(M_i)_{i \in I}$ together with BTZ-embeddings $\phi_{ij} : M_i \rightarrow M_j$ whenever it exists. The direct limit of this family is

$$\overline{M}_0 = \varinjlim_{i \in I} M_i = \left(\coprod_{i \in I} M_i \right) / \sim$$

where $x \sim y \Leftrightarrow \exists(i, j), \phi_{ij}(x) = y$.

It remains to check that the topology of such a limit is second countable. The proof is an adaptation of the arguments of Geroch given in [Ger68].

3.3 A remarks on Cauchy and BTZ-extensions

One may ask what happens if one takes the Cauchy-extension then the BTZ-extension. Is the resulting manifold Cauchy-maximal? The answer is no as the following example shows.

Example 3.12. Let $M_0 = \{\tau < 0, r > 0\}$ be the past half tube in cylindrical coordinates of $\mathbb{E}_0^{1,2}$ and let $p = (\tau = 0, r = 0)$. The spacetime M_0 is regular and globally hyperbolic. Let M_1 be its maximal Cauchy-extension, M_2 the maximal BTZ-extension of M_1 and M_3 the maximal Cauchy-extension of M_2 .

- $M_0 = \text{Reg}(\mathbb{E}_0^{1,2}) \setminus J^+(\{\tau = 0\})$
- $M_1 = \text{Reg}(\mathbb{E}_0^{1,2}) \setminus J^+(p)$.
- $M_2 = \mathbb{E}_0^{1,2} \setminus J^+(p)$
- $M_3 = \mathbb{E}_0^{1,2}$

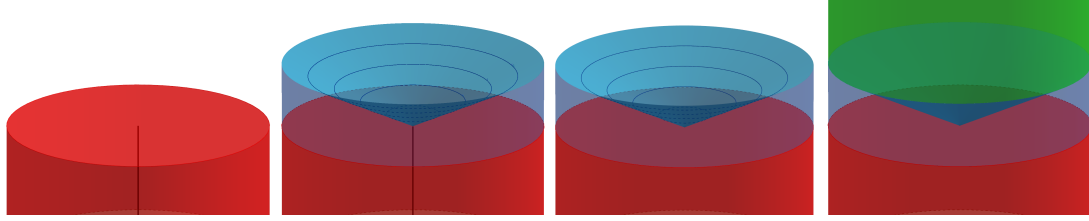


Figure 5: Successive maximal Cauchy-extension and BTZ-extension of a past half tube in $\mathbb{E}_0^{1,2}$. In red the initial half tube, in black the BTZ line missing. In blue its Cauchy-extension then the BTZ line is caught via the BTZ-extension. In green the final Cauchy-extension

Conjecture 1. Let M_0 be a globally hyperbolic singular manifold, M_1 its maximal Cauchy-extension, M_2 the maximal BTZ-extension of M_1 , M_3 the maximal Cauchy-extension of M_2 .

Then M_3 is both Cauchy-maximal and BTZ-maximal.

4 Cauchy-completeness and extensions of spacetimes

Is the Cauchy-completeness of a space-time equivalent to the Cauchy-completeness of the maximal BTZ-extension? The answer is yes and the whole section is devoted to the proof of this answer. We then aim at proving the following theorem.

Theorem III (Cauchy-completeness Conservation). Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold without BTZ point, the following are equivalent.

- (i) M is Cauchy-complete and Cauchy-maximal.
- (ii) There exists a Cauchy-complete and Cauchy-maximal BTZ-extension of M .

(iii) The maximal BTZ-extension of M is Cauchy-complete and Cauchy-maximal.

The proof decomposes into four parts. When taking a BTZ-extension, the Cauchy-surface changes. The proof of the theorem needs to modify Cauchy-surfaces in a controlled fashion. The first part is devoted to some lemmas useful to construct good spacelike surfaces. The other two parts solve the causal issues proving that the surfaces constructed using the first part are indeed Cauchy-surfaces. Pieces are put together in the fourth part to prove the theorem.

4.1 Surgery of Cauchy-surfaces around a BTZ-line

We begin by an example illustrating the situation we will soon manage.

Example 4.1. Consider $\mathbb{E}_0^{1,2}$ endowed with its coordinates (τ, r, θ) and the Cauchy-surface $\Sigma := \{\tau = 1\}$. The regular part of Σ , $\Sigma^* := \text{Reg}(\Sigma)$, is not a Cauchy-surface of the regular part of $\mathbb{E}_0^{1,2}$ since its Cauchy development is $\text{Reg}(\mathbb{E}_0^{1,2}) \setminus J^+(\{\tau = 1, r = 0\})$. The problem is that a curve such as $\{\tau = 2r + \tau_0, \theta = \theta_0\}$ is causal, inextendible in $\text{Reg}(\mathbb{E}_0^{1,2})$ and doesn't intersect Σ^* for $\tau_0 > 0$. A solution consists in noticing that Σ^* coincides with $\mathbb{H}_0^2 := \left\{ \tau = \frac{1+r^2}{2r} \right\}$ on $\{r = 1, \tau = 1\}$. Therefore, we can glue the piece of \mathbb{H}_0^2 inside the tube of radius 1 with the plane $\{\tau = 1\}$ outside the tube of radius 1 and get a complete Cauchy-surface Σ_1 of the regular part of $\mathbb{E}_0^{1,2}$. See figure 6 below.

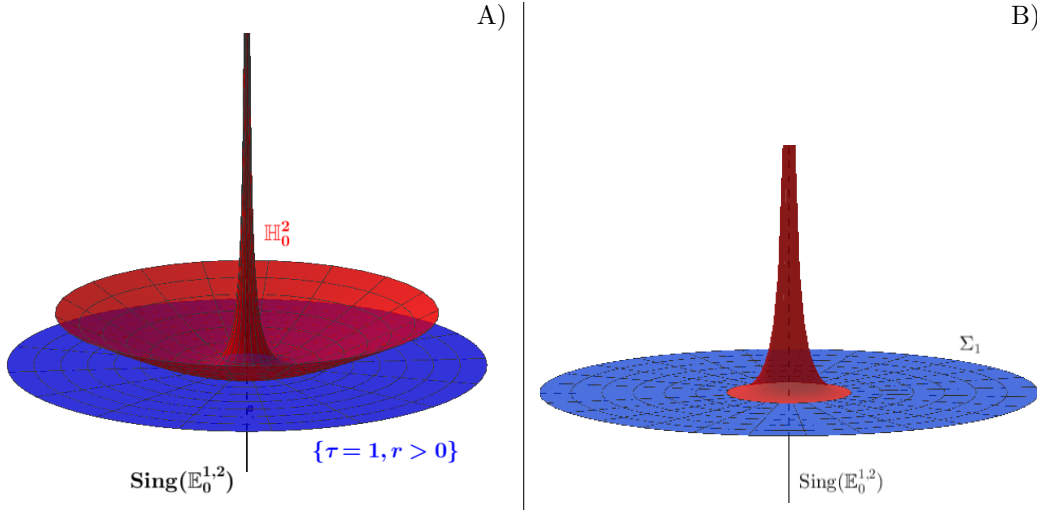


Figure 6: Two acausal surfaces, the boundary of their Cauchy development, two different gluings.

A) The blue plane represents the surface $\Sigma^* = \{\tau = 1, r > 0\}$ and the red surface is \mathbb{H}_0^2 .

B) The gluing Σ_1 of $\mathbb{H}_0^2 \cap \{r \leq 1\}$ with $\Sigma^* \cap \{r \geq 1\}$. It is a Cauchy-surface of $\mathbb{E}_0^{1,2}$.

Let M be a Cauchy-complete spacetime. Starting from a complete Cauchy-surface Σ of M , we need construct a complete Cauchy-surface of $M \setminus \Delta$ where Δ is a BTZ line. This is done locally around the singular line : the intersection of Σ with the boundary of a tube neighborhood of Δ gives a curves and the second point of the main Lemma 4.3 below show that such a curve can be extended to a complete surface avoiding the singular line of $\mathbb{E}_0^{1,2}$. This procedure is the heart of the proof of (ii) \Rightarrow (i) in Theorem III.

To obtain (i) \Rightarrow (iii), half of the work consists in doing the opposite task. Let M be a Cauchy-complete spacetime. Starting from a complete Cauchy-surface of M , we construct a complete Cauchy-surface of its maximal BTZ extension by modifying locally a Cauchy-surface of M around a singular line. We start from the intersection of the Cauchy-surface of M along the boundary of a tube around a singular line, this gives us a curve on a boundary of a tube in $\mathbb{E}_0^{1,2}$. The first point of the main Lemma 4.3 below show that such a curve can be extended to a complete surface which cuts the singular line of $\mathbb{E}_0^{1,2}$.

Definition 4.2. Define \mathbb{D}_R the Euclidean disc of radius $R > 0$ and \mathbb{D}_R^* the punctured Euclidean disc of radius $R > 0$.

We identify frequently \mathbb{D}_R with its embedding in $\{\tau = 0, r \leq R\}$.

Lemma 4.3. Let \mathcal{T} be a closed future half-tube in $\mathbb{E}_0^{1,2}$ of radius R and let $\tau_\Sigma^R : \partial\mathbb{D}_R \rightarrow \mathbb{R}_+$ be a smooth function. Then

- (i) there exists a piecewise smooth function $\tau_\Sigma : \mathbb{D}_R \rightarrow \mathbb{R}_+$ extending τ_Σ^R which graph is acausal, spacelike and complete;
- (ii) there exists a piecewise smooth function $\tau_\Sigma : \mathbb{D}_R^* \rightarrow \mathbb{R}_+$ extending τ_Σ^R which graph is acausal, spacelike and complete.

Before proving Lemma 4.3, we need to do some local analysis in a tube of $\mathbb{E}_0^{1,2}$. We begin by a local condition for acausality.

Lemma 4.4. Let $R > 0$ and let $\mathcal{T} := \{\tau > 0, r \leq R\}$ be a **closed** future half tube in $\mathbb{E}_0^{1,2}$ of radius R in cylindrical coordinates. Let $\tau_\Sigma \in \mathcal{C}^1([0, R] \times \mathbb{R}/2\pi\mathbb{Z}, \mathbb{R}_+^*)$ and $\Sigma = \text{Graph}(\tau_\Sigma) = \{(\tau_\Sigma(r, \theta), r, \theta) : (r, \theta) \in \mathbb{D}^*\}$, then (i) \Leftrightarrow (ii) \Leftrightarrow (iii).

- (i) Σ is spacelike and acausal
- (ii) Σ is spacelike

$$(iii) \quad 1 - 2\frac{\partial\tau_\Sigma}{\partial r} - \left(\frac{1}{r}\frac{\partial\tau_\Sigma}{\partial\theta}\right)^2 > 0$$

Proof. Beware that spacelike is a local condition but acausal is a global one. The implication (i) \Rightarrow (ii) is obvious.

Writing $\delta = \left(1 - 2\frac{\partial\tau_\Sigma}{\partial r} - \left(\frac{1}{r}\frac{\partial\tau_\Sigma}{\partial\theta}\right)^2\right)$ from direct computations :

$$ds_\Sigma^2 = \delta dr^2 + \left(\frac{1}{r}\frac{\partial\tau_\Sigma}{\partial\theta}dr - r d\theta\right)^2.$$

and let $c(s) = \tau_\Sigma(r(s), \theta(s))$, $s \in]s_*, s^*[$ be some path on Σ , then :

$$\frac{ds_c^2}{ds^2} = (r')^2 \delta(s) + \left(\frac{\partial\tau_\Sigma}{\partial\theta} \frac{r'}{r} - r\theta'\right)^2$$

Σ is spacelike iff its Riemann metric is postive definite iff $\delta > 0$ thus (ii) \Leftrightarrow (iii).

To prove (iii) \Rightarrow (i) take a smooth future causal curve $c = (\tau, r, \theta)$ such that $c(0) \in \Sigma$, i.e. $\tau_\Sigma(r(0), \theta(0)) = \tau(0)$. Since τ' is increasing, reparametrizing c if necessary, we can assume $\tau' > 0$. Let $f : s \mapsto \tau(s) - \tau_\Sigma(r(s), \theta(s))$ so that $f(s) = 0$ if and only if $c(s) \in \Sigma$, notice $f(0) = 0$. On the one hand, since c is causal, we have

$$r' \geq 0 \quad \text{and} \quad 2\tau'(s)r'(s) \geq (r')^2 + r^2(\theta')^2.$$

On the other hand, using $\delta > 0$, if $r' > 0$

$$2 \left(\frac{d}{ds} \tau_\Sigma(r, \theta) \right) r' = 2 \left(\frac{\partial \tau_\Sigma}{\partial \theta} \theta' + \frac{\partial \tau_\Sigma}{\partial r} r' \right) r' \quad (7)$$

$$< 2 \frac{\partial \tau_\Sigma}{\partial \theta} \theta' r' + (r')^2 - \left(\frac{r'}{r} \frac{\partial \tau_\Sigma}{\partial \theta} \right)^2 \quad (8)$$

$$\leq - \frac{(2\theta' r')^2 - 4 \left(-\frac{(r')^2}{r^2} \right) (r')^2}{-4 \frac{(r')^2}{r^2}} \quad (9)$$

$$= r^2 (\theta')^2 + (r')^2. \quad (10)$$

Let $s \in \mathbb{R}$, if $r'(s) > 0$ then the computation above shows $f'(s) > 0$. If $r'(s) = 0$ then $\theta'(s) = 0$ thus $f'(s) = \tau'(s) > 0$. Thus f is increasing, thus injective. Finally f cannot be naught twice and c cannot intersect Σ twice. Σ is thus acausal. \square

Lemma 4.5 (Completeness criteria). *Using the same notation as in Lemma 4.4 we have :*

1. Σ is spacelike and complete if

$$\exists C > 0, \forall (r, \theta) \in \mathbb{D}_R^*,$$

$$1 - 2 \frac{\partial \tau_\Sigma}{\partial r} - \left(\frac{1}{r} \frac{\partial \tau_\Sigma}{\partial \theta} \right)^2 \geq \frac{C^2}{r^2}$$

Furthermore, in this case the Cauchy development of Σ is $\mathcal{T} \setminus \Delta$.

2. If Σ is spacelike and complete then,

$$\lim_{(r, \theta) \rightarrow 0} \tau_\Sigma(r, \theta) = +\infty$$

Proof. We use the same notations as in the proof of Lemma 4.4. We insist on the fact that \mathcal{T} is **closed**, which means for instance that Σ has a boundary parametrised by $\partial \mathbb{D}_R$. It also means that a curve ending on the boundary of \mathcal{T} can be extended since it has an ending point.

1. Let $C > 0$ be such as $\delta > \frac{C^2}{r^2}$. It suffice to prove that a finite length curve in Σ is extendible. Let $\gamma : \mathbb{R} \rightarrow \Sigma$ be a finite length piecewise smooth curve on Σ . Write $\gamma(s) = (\tau_\Sigma(r(s), \theta(s)), r(s), \theta(s))$ for $s \in \mathbb{R}$ and $l(\gamma)$ its length. Since $l(\gamma) \geq \int_{\mathbb{R}} |r'(s)| \frac{C}{R} ds$ and $l(\gamma) < +\infty$, then $r' \in L^1$ and r converges as $s \rightarrow +\infty$, let $r_\infty := \lim_{s \rightarrow +\infty} r(s)$.

For all $a \in \mathbb{R}$, $l(\gamma) \geq \left| \int_0^a \frac{C|r'(s)|}{r(s)} ds \right| \geq C \left| \ln \left(\frac{r(0)}{r(a)} \right) \right|$. Thus

$$\forall a \in \mathbb{R}, \quad r(a) \geq r(0) e^{-\frac{1}{C} l(\gamma)} > 0$$

and thus $r_\infty > 0$.

Take $A > 0$ such as $\forall s \geq A, r(s) \in [r_*, r^*]$ with $r_* = r_\infty/2$ and $r^* = (r_\infty + R)/2$ then for all $b \geq a \geq A$:

$$l(\gamma) \geq \int_{[a, b]} r \left| \frac{\partial \tau}{\partial \theta} \frac{r'}{r^2} - \theta' \right| \quad (11)$$

$$\geq \int_{[a, b]} r_* \left| \frac{\partial \tau}{\partial \theta} \frac{r'}{r^2} - \theta' \right| \quad (12)$$

$$\geq \int_{[a, b]} r_* \left(|\theta'| - \left| \frac{\partial \tau}{\partial \theta} \frac{r'}{r^2} \right| \right) \quad (13)$$

$$\geq r_* \int_{[a, b]} |\theta'| - r_* \left(\max_{(r, \theta) \in [r_*, r^*] \times \mathbb{R}/2\pi\mathbb{Z}} \left| \frac{\partial \tau}{\partial \theta} \right| \right) \int_{[a, b]} \left| \frac{r'}{r^2} \right| \quad (14)$$

By integration by part, noting F a primitive of $|r'|$, and for some constant $C' > 0$.

$$l(\gamma) \geq r_* \int_{[a,b]} |\theta'| - C' \left(\left[\frac{F}{r^2} \right]_a^b + 2 \int_a^b \frac{F r'}{r^3} \right)$$

Since $\int_{\mathbb{R}} |r'| < +\infty$, F is bounded and can be chosen positive. Set $B = \sup_{s \in \mathbb{R}} R(s)$, thus for some $C''', C'''' > 0$,

$$\forall b > A, \quad l(\gamma) + C'' \geq C''' \int_{[a,b]} |\theta'|$$

Which proves that $\int_{[a,+\infty[} |\theta'| ds < +\infty$, so that $\theta(s)$ converges as $s \rightarrow +\infty$. Consequently, $\tau(r, \theta)$ converges in Σ . Since \mathcal{T} is **closed**, the curve γ is then extendible. We conclude that Σ is complete.

Let $c = (\tau_c, r_c, \theta_c) : \mathbb{R} \rightarrow \mathcal{T} \setminus \Delta$ be an inextendible future oriented causal curve. We must show that c intersects Σ . Since c is future oriented, τ_c is increasing and r_c is non-decreasing. Both functions have then limits at $\pm\infty$. Let $r^* = \lim_{s \rightarrow +\infty} r_c(s)$, $r_* = \lim_{s \rightarrow -\infty} r_c(s)$, $\tau_* = \lim_{s \rightarrow -\infty} \tau_c(s)$ and $\tau^* = \lim_{s \rightarrow +\infty} \tau_c(s)$. Since r_c is non-decreasing, $r^* > 0$ and since τ_c is increasing, $\tau_c' > 0$. Assume $\tau^* < \infty$, then $\tau_c' \in L^1([0, \infty[)$. We have on $[0, +\infty[$:

$$(r_c')^2 + r_c^2 (\theta')^2 - 2r_c' \tau_c' \leq 0 \quad (15)$$

$$(\theta')^2 \leq \frac{(\tau_c')^2 - (\tau_c' - r_c')^2}{r_c^2} \quad (16)$$

$$|\theta'| \leq \frac{1}{r_c(0)} \tau_c' \quad (17)$$

Thus $\theta' \in L^1([0, +\infty[)$ and θ has a limit at $+\infty$. The same way, we have :

$$|r_c' - \tau_c'| \leq \tau_c'$$

Thus $(r_c' - \tau_c') \in L^1([0, \infty[)$ and so is r_c' . Since r has a non zero limit at $+\infty$ and \mathcal{T} is **closed**, c is extendible ; therefore, $\tau^* = +\infty$.

Since $r^* \in]0, R]$ and since $\tau^* = +\infty$,

$$\exists s_0 \in \mathbb{R}, \forall s > s_0, \quad \tau_c(s) > \left(\max_{[r^*/2, r^*] \times \mathbb{R}/2\pi\mathbb{Z}} \tau_\Sigma \right) \geq \tau_\Sigma(r_c(s), \theta_c(s))$$

Similar arguments can be used to prove that either $\tau_* = 0$ or $r_* = 0$. Furthermore, one may check that the assumption implies that $\lim_{r \rightarrow 0} (\min_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \tau_\Sigma(r, \theta)) = +\infty$. This implies that $\min \tau_\Sigma > 0$.

Assume $\tau_* = 0$, since $\min \tau_\Sigma > 0$, we have :

$$\exists s_0 \in \mathbb{R}, \forall s < s_0, \quad \tau_c(s) < \min \tau_\Sigma \leq \tau_\Sigma(r_c(s), \theta_c(s))$$

If on the contrary we assume $\tau_* > 0$ and $r_* = 0$ then

$$\exists r \in \mathbb{R}_+, \min_{[0, r] \times \mathbb{R}/2\pi\mathbb{Z}} \tau_\Sigma > \tau_*$$

For such an $r \in \mathbb{R}_+$,

$$\exists s_0 \in \mathbb{R}, \forall s < s_0, \quad \tau_c(s) < \tau_\Sigma(r_c(s), \theta_c(s))$$

In any case, by continuity, there exists $s \in \mathbb{R}$ such that $\tau_c(s) = \tau(r_c(s), \theta_c(s))$ and thus such that $c(s) \in \Sigma$.

2. Since Σ is spacelike the point (ii) of Lemma 4.4 ensures that

$$1 - 2 \frac{\partial \tau_\Sigma}{\partial r} - \left(\frac{1}{r} \frac{\partial \tau_\Sigma}{\partial \theta} \right)^2 \geq 0$$

on $]0, R] \times \mathbb{R}/2\pi\mathbb{Z}$. Consider a sequence $(r_n, \theta_n) \rightarrow 0$, we assume $r_{n+1} < \frac{1}{2}r_n$, one can construct an inextendible piecewise continuously differentiable curve $c = (\tau_c, r_c, \theta_c) :]0, R] \rightarrow \Sigma$ such that

- $\forall s \in]0, R], r_c(s) = s$
- $\forall n \in \mathbb{N}, \theta_c(r_n) = \theta_n$
- $\forall r \in]0, R], |\theta'_c(r)| \leq \frac{2}{r_n}$

Writing $l(c)$ the length of c , we have :

$$l(c) = \int_0^R \sqrt{1 + r^2 \theta'_c(r)^2 - 2\tau'_c(r)} \quad (18)$$

$$\leq \int_0^R \sqrt{5 - 2\tau'_c(r)} \quad (19)$$

The integrand is well defined since $1 + r^2 \theta'_c(r)^2 - 2\tau'_c(r) > 0$. We deduce in particular that $\tau'_c \leq 5/2$ and thus $-\tau'_c \geq |\tau'_c| - 5$. By completeness of Σ , the length $l(c)$ of c is infinite thus $\int_0^R \sqrt{|\tau'_c|} = +\infty$ and thus $\int_0^{+\infty} |\tau'_c| = +\infty$. Finally,

$$\lim_{n \rightarrow +\infty} \tau_\Sigma(r_n, \theta_n) = \lim_{r \rightarrow 0} \tau_c(r) = \int_0^R (-\tau'_c) + \tau(R) \geq \int_0^R (|\tau'_c| - 5) + \tau(R) = +\infty$$

□

Proof of Lemma 4.3.

- (i) Define $\tau_\Sigma(r, \theta) = \tau_\Sigma^R(\theta) + M \left(\frac{1}{r} - \frac{1}{R} \right)$ with $M = 1 + \max_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \left| \frac{\partial \tau_\Sigma^R}{\partial \theta} \right|^2$.

Then : $\frac{\partial \tau_\Sigma}{\partial \theta} = \frac{\partial \tau_\Sigma^R}{\partial \theta}$ and $\frac{\partial \tau_\Sigma}{\partial r} = -\frac{M}{r^2}$. So that :

$$\delta = 1 - \left(-\frac{M}{r^2} \right) - \frac{1}{r^2} \left(\frac{\partial \tau_\Sigma^R}{\partial \theta} \right)^2 \quad (20)$$

$$= 1 + \frac{M - \left(\frac{\partial \tau_\Sigma^R}{\partial \theta} \right)^2}{r^2} \quad (21)$$

$$= 1 + \frac{1 + \max_{\theta \in \mathbb{R}/2\pi\mathbb{Z}} \left| \frac{\partial \tau_\Sigma^R}{\partial \theta} \right|^2 - \left(\frac{\partial \tau_\Sigma^R}{\partial \theta} \right)^2}{r^2} \quad (22)$$

$$> \frac{1}{r^2} \quad (23)$$

Therefore, the surface $\Sigma := \text{Graph}(\tau_\Sigma)$ is spacelike and complete.

(ii) Define

$$\tau_\Sigma(r, \theta) = \begin{cases} \left(\frac{2r-R}{R}\right)^2 \tau_\Sigma^R(\theta) + M\left(\frac{1}{r} - \frac{1}{R}\right) & \text{If } r \in [R/2, R] \\ \frac{M}{R} & \text{If } r \in [0, R/2] \end{cases}$$

where M is big enough so that the causality condition is satisfied on $[R/2, R] \times \mathbb{R}/2\pi\mathbb{Z}$. The graph of τ_Σ is spacelike and compact. \square

4.2 Cauchy-completeness and de-BTZ-fication

We give ourselves M a globally hyperbolic $\mathbb{E}_A^{1,2}$ spacetime for some $A \in \mathbb{R}_+$. One can check that taking a BTZ line away doesn't destroy global hyperbolicity.

Remark 4.6. $M \setminus \text{Sing}_0$ is globally hyperbolic.

Proof. M is causal and so is $M \setminus \text{Sing}_0$. Let $p, q \in M \setminus \text{Sing}_0$, consider a future causal curve c from p to q in M . By Proposition 1.13, we have a decomposition $c = \Delta \cup c^0$. Then $p \in \Delta$ or $\Delta = \emptyset$ and since $p \notin \text{Sing}_0$, $\Delta = \emptyset$, then $c \subset M \setminus \text{Sing}_0$. We deduce that the closed diamond from p to q in $M \setminus \text{Sing}_0$ is the same as the one in M . The latter is compact by global hyperbolicity of M , then so is the former. \square

We aim at proving the (ii) \Rightarrow (i) of Theorem III.

Proposition 4.7. *If M is Cauchy-complete and Cauchy-maximal then so is $M \setminus \text{Sing}_0$.*

A proof is divided into Propositions 4.8 and 4.9. The method consists in cutting a given complete Cauchy-surface which intersects the singular lines around each singular lines then use Lemma 4.3 to replace the taken away discs by a surface that avoids the singular line. We then check that the new surface is a Cauchy-surface and prove that the new manifold is Cauchy-maximal.

We assume M Cauchy-complete and Cauchy-maximal, write Σ a piecewise smooth spacelike and complete Cauchy-surface of M .

Proposition 4.8 (Cauchy-completeness). *$M \setminus \text{Sing}_0(M)$ is Cauchy-complete.*

Proof. Let $M' = M \setminus \text{Sing}_0(M)$. Let Δ be a BTZ-like singular line. We construct a complete Cauchy-surface Σ_2 of the complement of a Δ . The set of singular line being discrete, this construction extend easily to any number of singular line simultaneously.

From Proposition 2.7 there exists a neighborhood of $\Sigma \cap \Delta$ isometric to a future half tube $\mathcal{T} = \{\tau > 0, r \leq R\}$ of radius $R \in \mathbb{R}_+^*$ such that $\Sigma \cap \partial\mathcal{T}$ is an embedded circle. Let $\mathcal{T} \cap \Sigma = \text{Graph}(\tau_\Sigma)$ with $\tau_\Sigma : [0, R] \times \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}_+^*$. From Lemma 4.3, there exists $\tau_{\Sigma_2} : \mathbb{D}_R^* \rightarrow \mathbb{R}_+^*$ such that $\tau_{\Sigma_2} = \tau_\Sigma$ on $\partial\mathbb{D}_R$ and $\text{Graph}(\tau_{\Sigma_2})$ is acausal, spacelike and complete and furthermore, the Cauchy development $D(\text{Graph}(\tau_{\Sigma_2})) = \text{Reg}(\mathcal{T})$. Let Σ_2 the surface obtained gluing $\Sigma \setminus \mathcal{T}$ and $\text{Graph}(\tau_{\Sigma_2})$ along $\Sigma \cap \partial\mathcal{T}$. Since Σ and $\text{Graph}(\tau_{\Sigma_2})$ are spacelike and complete then so is Σ_2 .

We now show Σ_2 is a Cauchy-surface of $M \setminus \text{Sing}_0(M)$. Let c be an inextendible causal curve in M' , if $\inf(c) \in \text{Sing}_0(M)$ then one can extend it by adding the singular ray in its past to obtain an inextendible causal curve \bar{c} in M . The curve \bar{c} intersects Σ exactly once at some point $p \in \Sigma$.

- Assume $p \notin \mathcal{T}$, then $p \in \Sigma \setminus \mathcal{T} = \Sigma_2 \setminus \mathcal{T}$ and c intersects Σ_2 . Consider c_1 a connected component of $\bar{c} \cap \mathcal{T}$. Notice $D(\Sigma \cap \mathcal{T}) = \mathcal{T}$, thus c_1 is not inextendible in \mathcal{T} and thus c_1 leaves \mathcal{T} at some parameter s_1 . Then c_1 can be extended to

$$c_2 = c_1 \cup \{\tau > \tau_0, r = R, \theta = \theta_0\}$$

for $c_1(s_1) = (\tau_0, R, \theta_0)$, which is inextendible in \mathcal{T} . The curve c_2 thus intersects Σ , but since $c_1 \cap \Sigma = \emptyset$ then c_2 intersects Σ on the ray we added, and thus $\tau_\Sigma(R, \theta_0) > \tau_0$. The regular part of c_2 is inextendible in $\text{Reg}(\mathcal{T})$ and thus intersects Σ_2 exactly once and since Σ and Σ_2 agree on $\partial\mathcal{T}$ then $\text{Reg}(c_2)$ intersects Σ_2 on the added ray, and thus $c_1 \cap \Sigma_2 = \emptyset$. Finally, c intersects Σ_2 exactly once.

- Assume $p \in \mathcal{T}$, then consider c_1 the connected component of p in $\bar{c} \cap \mathcal{T}$. Either c_1 is inextendible or it leaves \mathcal{T} and can be extended by adding some ray $\{\tau > \tau_0, r = R, \theta = \theta_0\}$. Either way, write c_2 the inextendible extension of c_1 in \mathcal{T} . The regular part $\text{Reg}(c_2)$ is inextendible in $\text{reg}(\mathcal{T})$ and thus intersects $\Sigma_2 \cap \mathcal{T}$ exactly once. It cannot intersect Σ_2 on an eventually added ray $\{\tau > \tau_0, r = R, \theta = \theta_0\}$ other wise c_2 would intersect Σ twice. Then $\text{Reg}(c_2) \cap \Sigma_2 \in \text{Reg}(c_1) \subset c \cap \mathcal{T}$ and thus c intersects Σ_2 . The curve c cannot intersect Σ_2 outside \mathcal{T} thus every point of $c \cap \Sigma_2$ are in \mathcal{T} . Let c_3 another connected component of $\bar{c} \cap \mathcal{T}$. It cannot be inextendible otherwise it would intersect Σ , thus it leaves Σ and can be extended by adding some ray $\{\tau > \tau_0, r = R, \theta = \theta_0\}$, we obtain an inextendible curve c_4 . This curve intersects Σ and Σ_2 exactly once. Since $p \in c_1$ and $\bar{c} \cap \Sigma = \{p\}$, we have $c_3 \cap \Sigma = \emptyset$ and $c_4 \cap \Sigma \in \{\tau > \tau_0, r = R, \theta = \theta_0\}$. Therefore, $c_4 \cap \Sigma = c_4 \cap \Sigma_2 \neq \emptyset$ and, again, c_3 does not intersect Σ_2 .

Finally, c intersect Σ_2 exactly once.

Σ_2 is thus a Cauchy-surface of $M \setminus \text{Sing}_0(M)$.

□

Proposition 4.9 (Cauchy-maximality). *$M \setminus \text{Sing}_0$ is Cauchy-maximal.*

Proof. Write $M_0 = M \setminus \text{Sing}_0$, take M_1 a Cauchy-extension of M_0 and write $i : M_0 \rightarrow M$ the natural inclusion and $j : M_0 \rightarrow M_1$ the Cauchy-embedding. Consider $M_2 = (M \amalg M_1)/M_0$. Note $\pi : M \amalg M_1 \rightarrow M_2$ the natural projection, π is open. Assume M_2 is not Hausdorff. Let $(p, q) \in M \times M_1$ such that for all \mathcal{U} neighborhood of p and \mathcal{V} neighborhood of q , $\pi(\mathcal{U}) \cap \pi(\mathcal{V}) \neq \emptyset$. Take a sequence $(a_n)_{n \in \mathbb{N}} \in M_0^{\mathbb{N}}$ such that $\lim i(a_n) = p$ and $\lim j(a_n) = q$. Since $j \circ i^{-1} : M_0 \rightarrow j(M_0)$ is a $\mathbb{E}_A^{1,2}$ -isomorphism,

$$j \circ i^{-1}(I^+(p)) = j \circ i^{-1} \left[\text{Int} \left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} I^+(i(a_n)) \right) \right] \quad (24)$$

$$= \text{Int} \left(\bigcap_{N \in \mathbb{N}} \bigcup_{n \geq N} I^+(j(a_n)) \right) \quad (25)$$

$$= I^+(q) \cap j(M_0) \quad (26)$$

$$(27)$$

Consider a chart neighborhood \mathcal{U} of p and a chart neighborhood \mathcal{V} of q and assume $\mathcal{U} = \diamond_{p^-}^{p^+}$ and $\mathcal{V} = \diamond_{q^-}^{q^+}$. The image $j \circ i^{-1}(p^+)$ is in $I^+(q)$ thus $\diamond_{q^-}^{j \circ i^{-1}(p^+)}$ is a neighborhood of q and so is $\diamond_{q^-}^{j \circ i^{-1}(p^+)} \cap \diamond_{q^-}^{q^+}$. Then $I^+(q) \cap j(M_0) \cap \diamond_{q^-}^{j \circ i^{-1}(p^+)} \cap \diamond_{q^-}^{q^+} \neq \emptyset$ and we take some $a^+ \in M_0$ such that $i(a^+) \in I^+(p) \cap \mathcal{U}$ and $j(a^+) \in I^+(q) \cap \mathcal{V}$, so

$$\mathcal{U} \supset j \circ i^{-1} \left(\diamond_p^{i(a^+)} \right) = \diamond_q^{j(a^+)} \subset \mathcal{V}.$$

Then, from Proposition 1.19, \mathcal{U} and \mathcal{V} are in the same model space and p and q are of the same type. However, since $\text{Sing}_0(M_0) = \emptyset$ we also have $\text{Sing}_0(M_1) = \emptyset$, thus q is not a BTZ point and $p \in M_0$. Finally, $j \circ i^{-1}(p) = q$ and $\pi(p) = \pi(q)$. Therefore, M_2 is Hausdorff.

A causal analysis as in the proof of Proposition 4.8 shows M_2 is a Cauchy-extension of M . Since M is Cauchy-maximal, $M_2 = M$ and $M_1 = M_0$. Finally, M_0 is Cauchy-maximal \square

Proof of Proposition 4.7. Propositions 4.9 and 4.8 give respectively Cauchy-completeness and Cauchy-maximality of $M \setminus \text{Sing}_0(M)$. \square

4.3 Cauchy-completeness of BTZ-extensions

We prove that every BTZ-extension of a Cauchy-complete globally hyperbolic spacetime is Cauchy-maximal and Cauchy-complete. Let M_0 be a Cauchy-complete Cauchy-maximal globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold. We denote by M_1 the maximal BTZ-extension of M_0 and by M_2 the maximal Cauchy-extension of M_1 . We assume $M_0 \subset M_1 \subset M_2$ and take Σ_0 a Cauchy-surface of M_0 . The first step is to ensure that Σ_0 can be parametrised as a graph around a BTZ-line of M_1 .

Lemma 4.10. *Let Δ be a connected component of $\text{Sing}_0(M_2)$. For $R > 0$, write $\mathbb{D}_R = \{\tau = 0, r \leq R\}$ in $\mathbb{E}_0^{1,2}$.*

For all $p \in \Delta \cap (M_1 \setminus M_0)$, there exists \mathcal{U} a neighborhood of $]p, +\infty[$ such that

- *we have an isomorphism $\mathcal{D} : \mathcal{U} \rightarrow \mathcal{T} \subset \mathbb{E}_0^{1,2}$ with $\mathcal{T} = \{\tau \geq 0, r \leq R\}$ for some $R > 0$;*
- *we have a smooth function $\tau_{\Sigma_0} : \mathbb{D}_R^* \rightarrow \mathbb{R}_+$ such that*

$$\mathcal{D}(\Sigma_0 \cap \mathcal{U}) = \text{Graph}(\tau_{\Sigma_0})$$

and $\{\tau \leq \tau_{\Sigma_0}\} \subset M_0$.

Proof. From Proposition 2.7 the BTZ-line are complete in the future in M_2 and there are charts around future half of BTZ-lines in M_2 which are half tube of some constant radius. Consider such a tubular chart of radius R around a BTZ half-line Δ of M_2 which contains a point in $M_1 \setminus M_0$ and take a point $p \in \Delta \cap (M_1 \setminus M_0)$. We assume p has coordinate $\tau = 0$ and that $\mathcal{U} = \{-\tau^* < \tau < \tau^*, r \leq R\} \subset M_1$ for some $\tau^* > 0$ and that $\mathcal{V} = \{-\tau^* < \tau, r \leq R\} \subset M_2$. Consider future causal once broken geodesics defined on \mathbb{R}_+^* of the form

$$c_{\theta_0}(s) = \begin{cases} (s/2, s, \theta_0) & \text{if } s \leq R \\ (s/2, R, \theta_0) & \text{if } s > R \end{cases}$$

where $\theta_0 \in \mathbb{R}/2\pi\mathbb{Z}$. These curves parametrise the boundary of $J^+(p) \cap \mathcal{V}$. These curves are in the regular part of M_2 and start in M_0 . Each connected component of the intersection of these curve with M_0 is an inextendible causal curve. Take the first connected component, it intersects Σ_0 exactly once. Let B be the connected component of p in the boundary of $J^+(p) \cap \mathcal{V} \cap M_1$. Let $b : \mathbb{R}/2\pi\mathbb{Z} \rightarrow \Sigma_0 \cap B$ be the function $b : \theta \mapsto (\tau(\theta), r(\theta))$ which parametrizes $\Sigma \cap B$. We have B and Σ_0 are transverse since B is foliated by causal curves and Σ_0 is spacelike, thus $B \cap \Sigma_0$ is a topological 1-submanifold and b is continuous and bijective. Then b is a homeomorphism and the r coordinate on $B \cap \Sigma_0$ reaches a minimum $R' > 0$. In the tube $\{r \leq R', \tau > -\tau^*\}$, consider the future causal curves defined on \mathbb{R}_+^* ,

$$c_{r_0, \theta_0}(s) = \begin{cases} (s/2, s, \theta_0) & \text{if } s \leq r_0 \\ (s/2, r_0, \theta_0) & \text{if } s > r_0 \end{cases}$$

for $r_0 \in]0, R'[$ and $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. The intersection point with Σ_0 cannot be on the piece $s \in]0, r_0]$ while this piece is on a causal curve c_{θ_0} and that $r_0 < R'$. Thus, Σ_0 intersects all such curves

on the piece $s > r_0$ and the projection $\pi : \mathcal{V} \rightarrow \mathbb{D}^*$ restricted to $\Sigma \cap \mathcal{V}$ is continuous and bijective. we obtain a parametrisation of Σ_0 as the graph of some function $\tau_{\Sigma_0} : (r, \theta) \mapsto \tau_{\Sigma_0}$ in the tubular chart of radius R' .

$$\Sigma_0 = \text{Graph}(\tau_{\Sigma_0}) \quad \tau_{\Sigma_0}(r, \theta) > \frac{1}{2}r > 0$$

Since π is a projection along lightlike lines and Σ_0 is spacelike, τ_{Σ_0} is smooth. Furthermore, by definition of the curves c_{r_0, θ_0} the portion of curve before intersecting Σ_0 is in M_0 and thus we get a domain

$$\{r \in]0, R'], \theta \in \mathbb{R}/2\pi\mathbb{Z}, \tau \in]-\tau^*, \tau_{\Sigma_0}(r, \theta)]\}$$

included into M_0 .

□

Proposition 4.11. *M_1 is Cauchy-complete and Cauchy-maximal.*

Proof. The proof is divided into 3 steps. First we show that BTZ line in M_1 are complete in the future and future half of BTZ-lines of M_1 are contained in a tube neighborhood of some radius. Second we modify the smooth spacelike and complete Cauchy-surface Σ_0 of M_0 to obtain smooth spacelike and complete Cauchy-surface of M_1 . Third, we show that $(M_2 \setminus \text{Sing}_0(M_2))$ is a Cauchy-extension of M_0 and conclude.

Step 1 Consider a BTZ line Δ in M_2 . Consider \mathcal{T} a closed half-tube neighborhood of radius $R > 0$ around $]p, +\infty[$ for some $p \in \Delta \cap (M_1 \setminus M_0)$ given by Lemma 4.10. Write τ_{Σ_0} the parametrisation of Σ_0 by \mathbb{D}_R and $\mathcal{T}' = \text{Reg}(\mathcal{T}) = \mathcal{T} \setminus \Delta$. Consider the complement of M_0 in the half-tube \mathcal{T}' , subtract its future to M_0 and then add the full half-tube, namely:

$$M = \mathcal{T}' \cup (M_0 \setminus (J^+(\mathcal{T}' \setminus M_0))).$$

Since $\Sigma_0 \cap \mathcal{T} = \text{Graph}(\tau_{\Sigma_0})$ and $J_{\mathcal{T}'}^-(\Sigma_0) \subset M_0$, then $J^+(\mathcal{T}' \setminus M_0) \subset J^+(\Sigma_0)$ and thus $\Sigma_0 \subset M$. Let c be an future inextendible causal curve in M . Remark that by construction of M , the curve c cannot leave $\mathcal{T}' \setminus M_0$. Therefore, since c is connected, c decomposes into two connected consecutive parts : a M_0 part and then a part in $\mathcal{T}' \setminus M_0$.

- Assume $c \cap (\mathcal{T}' \setminus M_0) \neq \emptyset$. Since Σ_0 is spacelike and complete by Lemma 4.5 $\lim_{(r, \theta) \rightarrow 0} \tau_{\Sigma_0}(t, \theta) = +\infty$. The intermediate value theorem then ensures that c intersects $\Sigma_0 \cap \mathcal{T}'$. Furthermore, once in $\mathcal{T}' \setminus M_0$, the curve c stays in $\mathcal{T}' \setminus M_0$ thus $c \cap M_0$ is an inextendible causal curve of M_0 which intersects Σ_0 exactly once. Then, c intersects Σ_0 exactly once.
- Assume $c \cap (\mathcal{T}' \setminus M_0) = \emptyset$. The curve c is then a causal curve in M_0 and any inextendible extension of c in M_0 intersects Σ_0 exactly once. Such an inextendible extension cannot leaves $J^+(\mathcal{T}' \setminus M_0)$ once it enters it, therefore its intersection point with Σ_0 is on c .

Therefore Σ_0 is a Cauchy-surface of M , M is a Cauchy-extension of a neighborhood of Σ_0 in M_0 and by unicity of the maximal Cauchy-extension Theorem 2.2, M is a subset of M_0 . Finally, M_0 contains \mathcal{T}' and thus M_1 contains \mathcal{T} .

Step 2 Consider a BTZ line Δ in M_2 and a tube neighborhood \mathcal{T}_Δ of Δ given by Lemma 4.10. Let τ_{Σ_0} be the parametrisation of Σ_0 inside \mathcal{T}_Δ . From step one, \mathcal{T}_Δ is in M_1 thus from Lemma 4.3, one can extend $\Sigma_0 \cap \partial \mathcal{T}_\Delta$ to some smooth spacelike complete surface $\text{Graph}(\tau_\Delta)$ in \mathcal{T}_Δ parametrised by \mathbb{D}_R for some R . The number of BTZ-line being enumerable, one can choose the neighborhoods \mathcal{T} around each BTZ-line such that they don't intersect. Thus

this procedure can be done around every BTZ-line simultaneously. A causal discussion as in Proposition 4.8 shows that the surface

$$\Sigma_1 := (\Sigma_0 \setminus \bigcup_{\Delta} \mathcal{T}_{\Delta}) \cup \bigcup_{\Delta} \text{Graph}(\tau_{\Delta})$$

is a piecewise smooth spacelike and complete Cauchy-surface of M_1 . Therefore, M_1 is **Cauchy-complete**.

Step 3 Consider now $M = (M_2 \setminus \text{Sing}_0(M_2))$ and c a future inextendible causal curve c in M . The curve c can be extended to some c' inextendible curve of M_2 . From Lemma 4.3, c' decomposes into two connected consecutive part : Δ its BTZ part, then c^0 its non-BTZ part. By definition of M , $\Delta = c' \setminus c$ and $c^0 = c$. Since M_2 is a Cauchy-extension of M_1 , the curve c' intersects Σ_1 exactly once. On the one hand, Σ_1 and Σ_0 coincides outside the tubes \mathcal{T}_{Δ} . On the other hand, notice that an inextendible causal curve inside a \mathcal{T}_{Δ} intersects $\Sigma_0 \cap \mathcal{T}$ if and only if it intersects $\text{Graph}(\tau_{\Delta})$. Thus c' also intersects Σ_0 exactly once and thus c intersects Σ_0 exactly once. We deduce that M is a Cauchy-extension of M_0 and, by maximality of M_0 , we obtain $M = M_0$. Therefore, $M_2 = M_1$ and M_1 is **Cauchy-maximal**. □

4.4 Proof of the Main Theorem

Theorem III (Cauchy-completeness Conservation). *Let M be a globally hyperbolic $\mathbb{E}_A^{1,2}$ -manifold without BTZ point, the following are equivalent.*

- (i) M is Cauchy-complete and Cauchy-maximal.
- (ii) There exists a Cauchy-complete and Cauchy-maximal BTZ-extension of M .
- (iii) The maximal BTZ-extension of M is Cauchy-complete and Cauchy-maximal.

Proof. The implication (iii) \Rightarrow (ii) is obvious. The implication (ii) \Rightarrow (i) is given by Propositions 4.7. The implication (i) \Rightarrow (iii) is given by Proposition 4.11. □

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